Asymptotic Distributions and Expansions of Multivariate Maxima in Triangular Schemes

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Limiting dfs of maxima

Assume that $X$, $Y$ are rvs with joint df $H$ having $[-1, 0]$-uniform margins, $(X_i, Y_i)$ iid. Then,

$$H^n(x/n, y/n)$$

is the df of normalized maxima

$$\max(nX_1, \ldots, nX_n), \max(nY_1, \ldots, nY_n).$$

Goal: Find limiting df $G(x, y)$ of $H^n(x/n, y/n)$.


Pickands representation:

$$G(x, y) = \exp \left( (x + y)D \left( \frac{x}{x + y} \right) \right), \quad x, y \leq 0,$$

where $D$ is the Pickands-dependence function.
Ex. 1: (i) if $H_\rho$ is a normal df with fixed correlation coefficient $\rho \in [0, 1)$ and $[-1, 0]$-uniform margins, then independence holds, that is,

$$G(x, y) = \exp(x + y) \quad x, y \leq 0.$$ 

(ii) Generally, if $X$, $Y$ are tail-independent, that is,

$$P(Y > u | X > u) \to 0 \quad \text{for} \quad u \uparrow 0,$$

Then, independence holds.

To get non-trivial results we consider triangular arrays; e.g. normal dfs with correlation coefficients $\rho(n) \uparrow 1$.

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A spectral representation of dfs

The extreme value df $G$ can be written as

$$G(x, y) = \exp(cD(z)),$$

where $c = x + y$ and $z = x/(x + y)$ are the radial and angular coordinates.

Generally, a bivariate df $H$ with support in $(-\infty, 0]^2$ has such a representation in $c$ and $z$,

$$H(x, y) = H(c(z, 1 - z)).$$

Keeping $z \in [0, 1]$ fixed this allows to represent the bivariate df $H$ by a family of 1-dimensional dfs

$$H_z(c) = H(c(z, 1 - z)), \quad c \leq 0.$$

We make use of the densities

$$h_z = \frac{\partial H_z}{\partial c},$$

called the spectral density (see Falk et al. (2007)).
Conditions on the spectral density

The importance of the spectral density becomes evident by mentioning the fact that convergence of \( h_z(c) \), for \( c \uparrow 0 \), implies that

\[
h_z(c) \to D(z), \quad c \uparrow 0,
\]

where \( D \) is the Pickands-dependence function.

Frick and Reiss (2009) use a refined condition, namely,

\[
h_z(c) = D(z) + B_{\beta}(c)A(z) + o(B_{\beta}(c)), \quad c \uparrow 0 \quad (1)
\]

with a factorization in \( c \) and \( z \).
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A limit theorem

Th. 1 [Frick and Reiss (2010)] Including a link-function $F$ and depending on the sample size $n$ assume that

$$h_{\beta(n), z} \left( \frac{c}{n} \right) = zF \left( B_{\beta(n)} \left( \frac{c}{n} \right) \hat{A}(z) \right) + (1 - z)F \left( B_{\beta(n)} \left( \frac{c}{n} \right) \hat{A}(1 - z) \right) + o(1)$$

Then, one gets limiting dfs of maxima

$$G(x, y) = \exp \left( xF \left( \lambda \hat{A} \left( \frac{x}{x+y} \right) \right) + yF \left( \lambda \hat{A} \left( \frac{y}{x+y} \right) \right) \right).$$

depending on $\hat{A}$ and $F$. 
Special cases

We mention three special cases.

Ex. 2. (i) If \( F(u) = 1 + u \) on \([-1, 0]\) then condition (2) is closely related to condition (1).

(ii) We have asymptotic independence of marginal maxima if \( \lambda = \infty \).

(iii) [Hüsler and Reiss (1989)] Let \( H_{\rho(n)} \) be bivariate normal dfs with \([-1, 0]\)-uniform margins and correlation coefficients \( \rho_n \) such that

\[
(1 - \rho_n) \log(n) \to \lambda^2 \in [0, \infty], \quad n \to \infty. \tag{3}
\]

In that case \( F \) is the standard normal df \( \Phi \). One gets limiting dfs \( H_\lambda \) of maxima for \( \lambda \in [0, \infty] \) with independence if \( \lambda = \infty \).
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Expansions: the normal case


Th. 2. [Frick and Reiss (2013)] Let again $H_{\rho(n)}$ be the bivariate normal df with $[-1, 0]$-uniform margins as specified above. Assume that

$$\lim_{n \to \infty} (1 - \rho_n) \log(n) = \infty, \quad n \to \infty.$$  \hfill (4)

Then,

$$H^n_{\rho(n)}(x / n, y / n) =$$

$$\exp \left( x + y + \alpha(\rho, n)(xy)^{\frac{1}{1+\rho}} \right) \left( 1 + o(\alpha(\rho_n, n)) \right)$$

where

$$\alpha(\rho, n) = \frac{(1 + \rho)^{\frac{3}{2}}}{n^{\frac{1-\rho}{1+\rho}} (1 - \rho)^{\frac{1}{2}} (4\pi \log n)^{\frac{\rho}{1+\rho}}}.$$  

Imposing a condition on the spectral density one may deduce a general result about expansions.
Penultimate distributions: the normal case


Th. 3. [Frick and Reiss (2013)] The expansion in Th. 2 can be replaced by dfs $H_{\lambda(\rho(n))}$ (as given in Ex. 2) with $\lambda(\rho(n)) \uparrow \infty$ as $n \to \infty$. 
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Oberwolfach meeting, 1987