Asymptotic Distributions and Expansions of Multivariate Maxima in Triangular Schemes

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Overview

Overview

Limiting dfs of maxima

Assume that X, Y are rvs with joint df H having $[-1, 0]$-uniform margins, (X_i, Y_i) iid. Then,

$H^n(x/n, y/n)$ is the df of normalized maxima

$$\max(nX_1, \ldots, nX_n), \; \max(nY_1, \ldots, nY_n).$$

Goal: Find limiting df $G(x, y)$ of $H^n(x/n, y/n)$!

Bivariate Case: Tiago de Oliveira (1958), (1962/63), (1975), Geffroy (1958), Sibuya (1960)
Multivariate Case: de Haan and Resnick (1977), Deheuvels (1978), (1981), Pickands (1981)

Pickands representation:

$$G(x, y) = \exp\left((x+y)D\left(\frac{x}{x+y}\right)\right), \quad x, y \le 0,$$

where D is the Pickands-dependence function.

Why triangular arrays?

Ex. 1: (i) if H_ρ is a normal df with fixed correlation coefficient $\rho \in [0, 1)$ and $[-1, 0]$-uniform margins, then independence holds, that is,

$$G(x, y) = \exp(x + y) \quad x, y \leq 0 .$$

(ii) Generally, if $X,\ Y$ are tail-independent, that is,

$$P(Y > u | X > u) \to 0 \quad \text{for} \quad u \uparrow 0,$$

Then, independence holds.

To get non-trivial results we consider triangular arrays; e.g. normal dfs with correlation coefficients $\rho(n) \uparrow 1$.

Multivariate case: Hüsler and Reiss (1989), Joe (1994), Hashorva (2005) and (2006), Nikoloulopoulos et al. (2009), Frick and Reiss (2010), Padoan (2011)
Processes: Brown and Resnick (1977), Kabluchko (2011)

Overview

A spectral representation of dfs

The extreme value df G can be written as

$$G(x,y) = \exp(cD(z)),$$

where $c = x + y$ and $z = x/(x+y)$ are the radial and angular coordinates.

Generally, a bivariate df H with support in $(-\infty, 0]^2$ has such a representation in c and z,

$$H(x,y) = H\big(c(z, 1-z)\big).$$

Keeping $z \in [0,1]$ fixed this allows to represent the bivariate df H by a family of 1-dimensional dfs

$$H_z(c) = H(c(z, 1-z)), \quad c \le 0.$$

We make use of the densities

$$h_z = \frac{\partial H_z}{\partial c},$$

called the **spectral density** (see Falk et al. (2007)).

Conditions on the spectral density

The importance of the spectral density becomes evident by mentioning the fact that convergence of $h_z(c)$, for $c \uparrow 0$, implies that

$$h_z(c) \to D(z),\ c \uparrow 0,$$

where D is the Pickands-dependence function.

Frick and Reiss (2009) use a refined condition, namely,

$$h_z(c) = D(z) + B_\beta(c)A(z) + o(B_\beta(c)), \quad c \uparrow 0 \tag{1}$$

with a factorization in c and z.

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A limit theorem

Th. 1 [Frick and Reiss (2010)] Including a link-function F and depending on the sample size n assume that

$$h_{\beta(n),z}\left(\frac{c}{n}\right) = zF\left(B_{\beta(n)}\left(\frac{c}{n}\right)\hat{A}(z)\right) + (1-z)F\left(B_{\beta(n)}\left(\frac{c}{n}\right)\hat{A}(1-z)\right) + o(1) \tag{2}$$

Then, one gets limiting dfs of maxima

$$G(x,y) = \exp\left(xF\left(\lambda\hat{A}\left(\frac{x}{x+y}\right)\right) + yF\left(\lambda\hat{A}\left(\frac{y}{x+y}\right)\right)\right).$$

depending on $\hat{A}$ and F.

Special cases

We mention three special cases.

Ex. 2. (i) If $F(u) = 1 + u$ on $[-1, 0]$ then condition (2) is closely related to condition (1).

(ii) We have asymptotic independence of marginal maxima if $\lambda = \infty$.

(iii) [Hüsler and Reiss (1989)] Let $H_{\rho(n)}$ be bivariate normal dfs with $[-1, 0]$-uniform margins and correlation coefficients ρ_n such that

$$(1 - \rho_n)\log(n) \to \lambda^2 \in [0, \infty], \quad n \to \infty. \tag{3}$$

In that case F is the standard normal df Φ. One gets limiting dfs H_λ of maxima for $\lambda \in [0, \infty]$ with independence if $\lambda = \infty$.

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Expansions: the normal case

Literature: Reiss (1989), Ledford and Tawn (1996).

Th. 2. [Frick and Reiss (2013)] Let again $H_{\rho(n)}$ be the bivariate normal df with $[-1, 0]$-uniform margins as specified above. Assume that

$$(1-\rho_n)\log(n) \to \infty, \quad n \to \infty. \tag{4}$$

Then,

$$H^n_{\rho(n)}(x/n, y/n) = \exp\left(x + y + \alpha(\rho_n, n)(xy)^{\frac{1}{1+\rho_n}}\right)(1 + o(\alpha(\rho_n, n)))$$

where

$$\alpha(\rho, n) = \frac{(1+\rho)^{\frac{3}{2}}}{n^{\frac{1-\rho}{1+\rho}}(1-\rho)^{\frac{1}{2}}(4\pi\log n)^{\frac{\rho}{1+\rho}}}.$$

Imposing a condition on the spectral density one may deduce a general result about expansions.

Penultimate distributions: the normal case

Literature: Fisher and Tippett (1928), Gomes (1984), Reiss (1989), Kaufmann (2000), Reiss, Thomas and Kaufmann (2007).

Th. 3. [Frick and Reiss (2013)] The expansion in Th. 2 can be replaced by dfs $H_{\lambda(\rho(n))}$ (as given in Ex. 2) with $\lambda(\rho(n)) \uparrow \infty$ as $n \to \infty$.

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References

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