Bayesian nonparametrics for multivariate extremes including censored data

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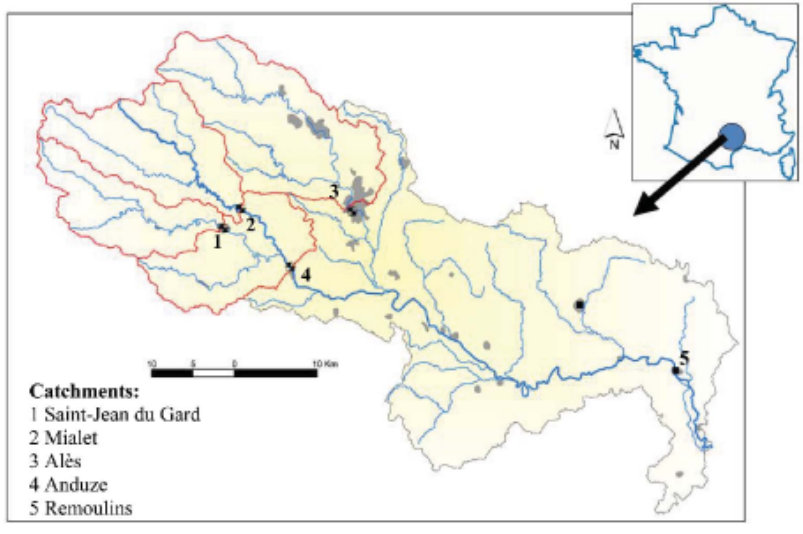
Censored Multivariate extremes: Why bother ?

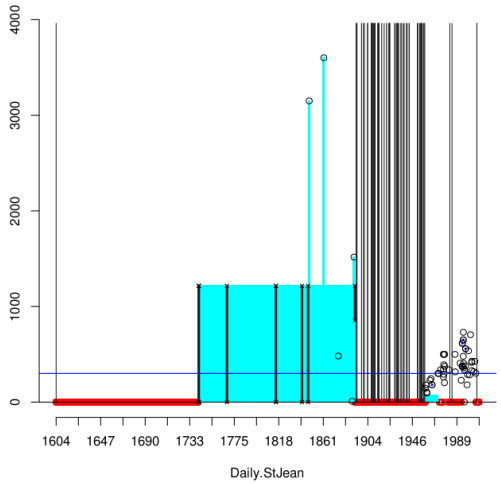
- ex: Daily streamflow at 4 neighbouring sites (Gardons, Cévennes, France).
- Return levels for jointly extreme events ?
- Recent, 'clean' series very short
- Censored historical data availble from archives, variable 'perception thresholds' for floods. Earliest: 1604.

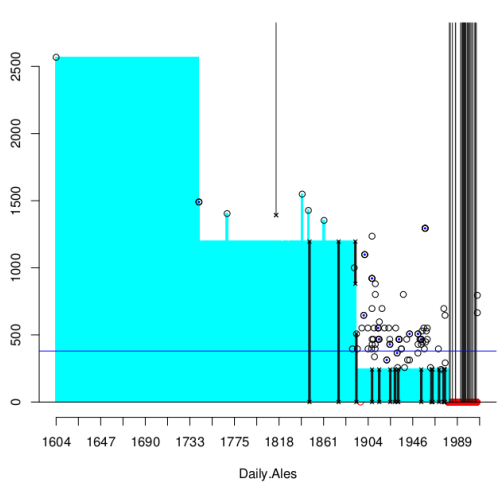


How to use all different kinds of data ?

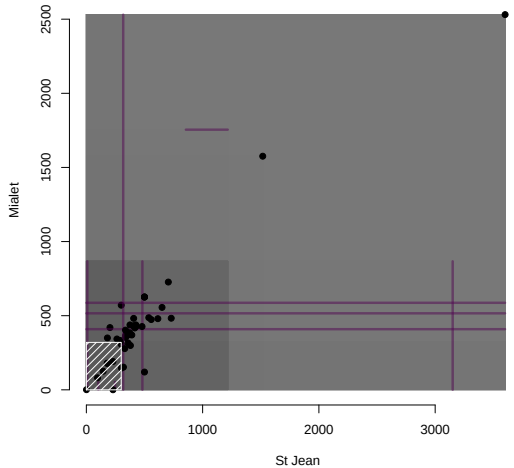
Censored data: pairwise plots

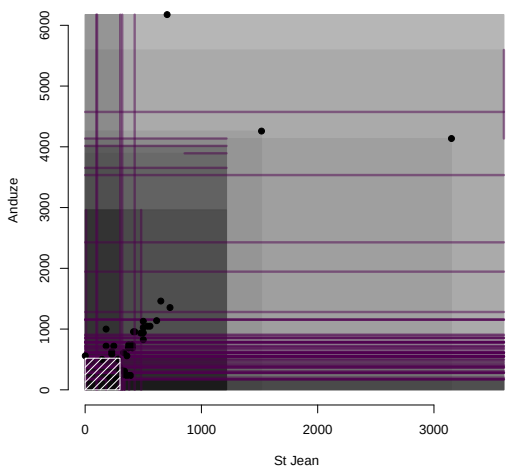
Univariate series:

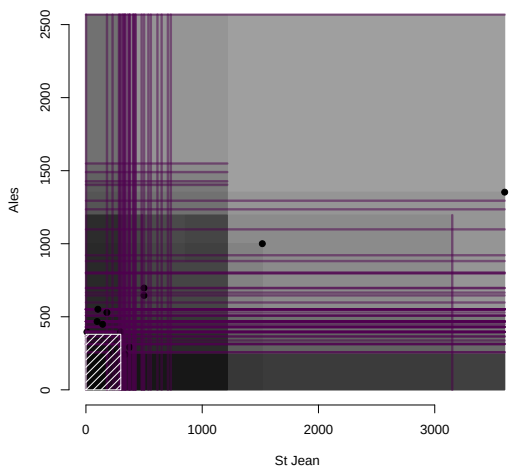




Bivariate:







Purposes

- Take into account as many data as possible
  $\rightarrow$ Censored likelihood, integration problems

- Estimate marginal GPD parameters + dependence structure jointly
  $\rightarrow$ additional Gibbs step in a MCMC algo.

- Probability of failure regions (return periods for jointly extreme events)
  $\rightarrow$ Model for excesses (POT-Poisson).

Approach: use Dirichlet consistency properties

- Angular measure model on the simplex.

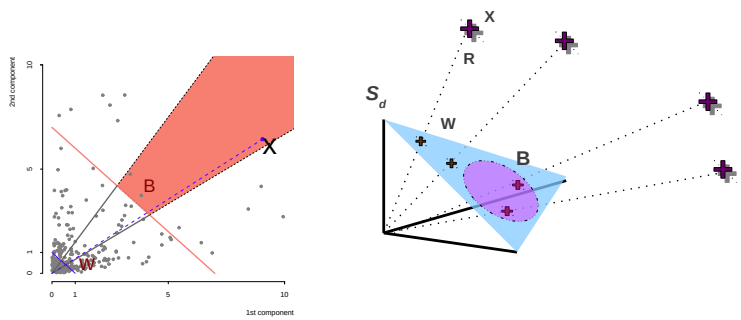
- Dirichlet distributions: Nice marginalization and conditioning properties.

- Dirichlet mixture model for angular measures
  Boldi & Davison, 2007 ; Sabourin & Naveau, 2013 :
  - Flexible, non parametric.
  - Reversible jumps - MCMC algorithm available.

Adaptation needed: censored likelihood and variable threshold
$\Rightarrow$ **Data augmentation**

Excess Data: polar decomposition

- Vectorial obs $\mathbf{Y}_t = (Y_{1,t}, \ldots, Y_{d,t})$; *i.i.d.*
- Margin $Y_{j,t} \sim F_j$: Pareto above threshold v_j.
- Dependence defined among the $X_{j,t} = -1/log\, F_{j,t}(Y_j, t)$: standard Fréchet
- Polar coordinates:
  $R = \|X\|_1 = \sum_j X_j$: 'radius' (norm), $\quad \mathbf{W} = \frac{\mathbf{X}}{R}$: 'angle'



$$\mathbf{W} \in \text{simplex } \mathbf{S}_d = \{(w_1, \ldots w_d) : \sum w_j = 1, w_j \geq 0\}\,.$$

Angular measure models

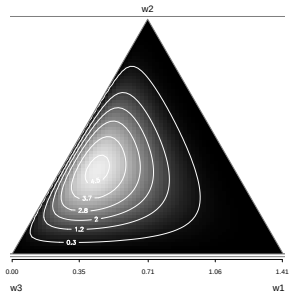
- Joint distribution of excesses: product measure in polar coordinates.
$$P(R > r, \mathbf{W} \in B) \propto \frac{1}{r} H(B)$$
- H: angular measure, distribution of angles. Non parametric: Valid *iff* $\int_{\mathbf{S}_d} w_i \, \mathrm{d}H(\mathbf{w}) = \frac{1}{d}$ $(1 \leq i \leq d)$

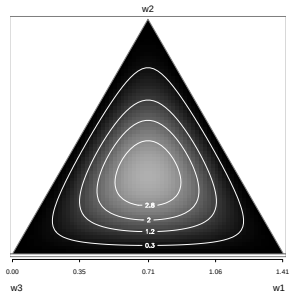
- *Inference*: **Empirically**, Einmahl et.al., 2001, Einmahl, Segers, 2009, Guillotte et al, 2011. : No explicit expression of asymptotic variance, Bayesian inference with $d = 2$ only.
Restriction to **parametric subclass** Gumbel, Coles&Tawn, Cooley *et.al.*. . . : logistic family, Pairwise-Beta . . .

- Preferred here: **Dirichlet Mixture model**: Compromise
  - Flexibility (weakly dense family)
  - Uncertainty quantification through parameters, Bayesian implementation.

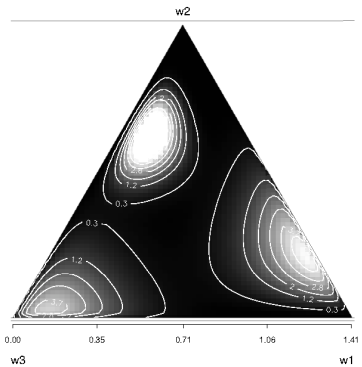
Dirichlet mixture model for angular measure

- Dirichlet distribution: generalization of Beta to higher dimensions.
- 1 location parameter (point on the simplex): 'center' + 1 concentration parameter.





- Dirichlet mixture model, k components: $h(\mathbf{w}) = \sum_{m=1}^{k} p_m \, \text{diri}_m(\mathbf{w})$. Boldi, Davison, 2007



- H valid $\Leftrightarrow$ center of mass of the location parameters = center of the simplex.

Issues: moments constraints and censored data

(i) Sampling the posterior distribution with MCMC methods. Constraints $\Rightarrow$ Sampling issues

- Re-parametrization: No more constraint, sampling is manageable for $d = 5$: **Sabourin, Naveau, 2013**

(ii) Inference with censored data:

- Variable threshold: changing normalizing constants.
- Censored data = segments or boxes. Integrating the likelihood (density $\frac{\mathrm{d}r}{r^2}\,\mathrm{d}H(\mathbf{w})$) on rectangles ?

Sabourin, ***under review*** ; **Sabourin**, **Renard**, ***in preparation***

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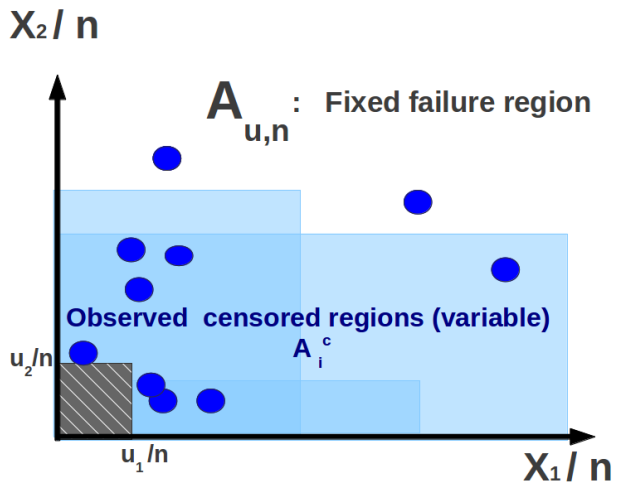
Poisson model for variable threshold

$$\left\{ \left(\frac{t}{n}, \frac{\mathbf{X}_t}{n} \right), \ 1 \leq t \leq n \right\} \sim \text{Poisson Process } (\text{Leb} \times \lambda) \quad \text{on } [0,1] \times A_{u,n}$$

λ: ' exponent measure', with Dirichlet Mixture angular component

$$\frac{\mathrm{d}\lambda}{\mathrm{d}r \times \mathrm{d}\mathbf{w}}(r, \mathbf{w}) = \frac{d}{r^2} h(\mathbf{w}) \, .$$



'A_i' data overlapping threshold: included in Poisson likelihood as

$$N \left\{ \left(\frac{t_2}{n} - \frac{t_1}{n} \right) \times \frac{1}{n} A_i \right\} = 0$$

'Censored' likelihood: model density integrated over boxes

- Ledford & Tawn, 1996: GEV parametric model, partially extreme data censored at threshold.
- Here: more general censoring framework, Poisson likelihood $\frac{\mathrm{d}\lambda}{\mathrm{d}x}$, non parametric, no explicit expression.
- Two terms without closed form:
  - Censored regions A_i overlapping threshold:

$$\exp\left\{-(t_2 - t_1)\lambda(A_i)\right\}$$

  - Classical censoring above threshold

$$\int_{\text{censored region}} \frac{\mathrm{d}\lambda}{\mathrm{d}\mathbf{x}}\,.$$

Data augmentation

One more Gibbs step, no more numerical integration.

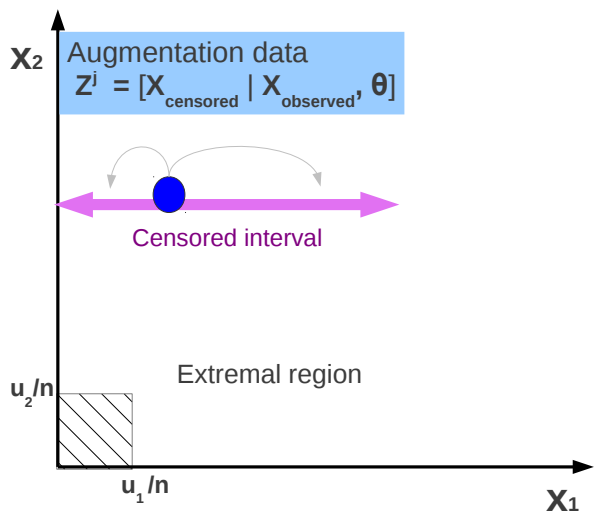
- Objective: sample $[\theta|Obs]$. (Parameter space: Θ)
- Additional variables (replace missing data component): $\mathcal{Z}$
- Full conditionals $[Z_i|Z_{j\neq j}, \theta, Obs]$ explicit (Thanks Dirichlet): $\rightarrow$ Gibbs sampling.
- Sample $[z, \theta|Obs]_+$ (augmented distribution) on $\Theta \times \mathcal{Z}$.

Censored regions above threshold

$$\int_{\text{Censored region}} \frac{d\lambda}{dx}\, dx_{j_1:j_r} \quad :$$

Generate missing components under univariate conditional distributions

$$\mathbf{Z}^j_{1:r} \sim [X_{\text{missing}} | X_{\text{obs}}, \theta]$$

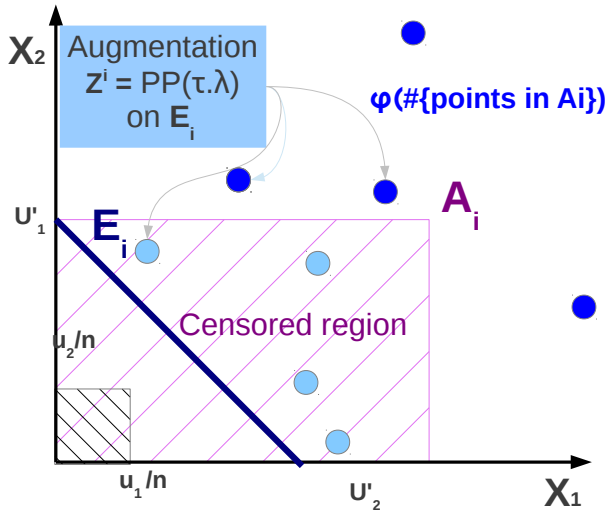


Dirichlet $\Rightarrow$ Explicit univariate conditionals

Exact sampling of censored data on censored interval

Censored regions overlapping threshold

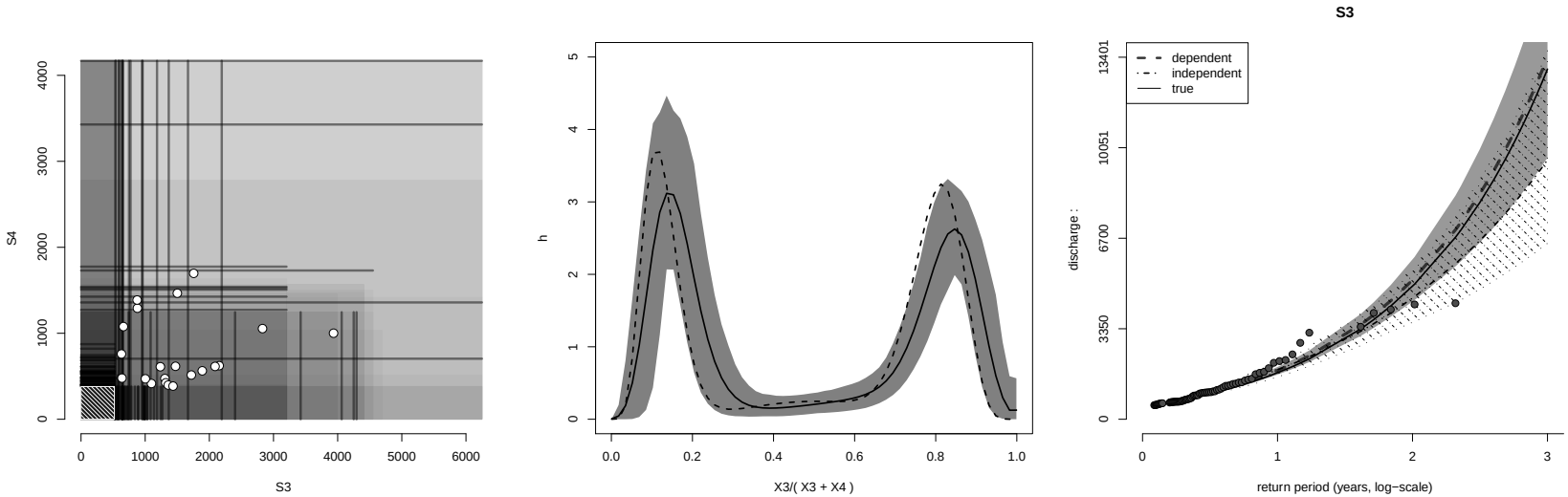
$$e^{-(t_{2,i}-t_{1,i})\lambda(A_i)} \Leftrightarrow \begin{cases} \text{augmentation Poisson process } N_i \text{ on } E_i \supset A_i. \\ + \\ \text{Functional } \varphi(N_i) \end{cases}$$



$$[z, \theta | Obs] \propto \underbrace{\ldots}_{\text{density terms, prior, augmented missing components}} [N_i]\varphi(N_i)$$

Results on simulated data

- Angular measure: Dirichlet, $d = 4$, $k = 3$ mixture components
- Censoring: same pattern as real data.



Pair $(S3, S4)$: data, predictive *a posteriori* of the angular measure, quantile curve for $S3$.

Conclusion

- Accounting for all types of censored data: manageable using Dirichlet consistency properties (marignalization, conditioning)

- High dimension (GCM grid, spatial fields)?
  - Impose a reasonable structure (sparse) on Dirichlet parameters
  - Dirichlet Process ? Challenges :
    Discrete random measure $\neq$ continuous framework for GPP's.

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Details: Augmentation Poisson process

$$e^{-(t_{2,i}-t_{1,i})\lambda(A_i)} \text{ ??}$$

Proposal $\mathbf{Z}^i \sim PP(\tilde{\lambda}^i)$; define f^i s.t.

$$\mathbb{E}\left\{e^{-\sum_j f^i(\mathbf{Z}^i{}_j)}\right\} = \mathcal{L}\mathrm{apl}_{\mathbf{Z}^i}(f^i) = e^{-\int 1-e^{-f}\,\mathrm{d}\tilde{\lambda}^i} = e^{-n_i\lambda(A_i)}$$

$$[\mathbf{Z},\theta|\mathbf{O}]_+ \;\propto\; [\theta]\prod_j\left\{[\mathbf{Z}^{\mathbf{j}}_{\mathbf{1:r}}|\mathbf{O},\theta]\right\}\prod_i\left\{[\mathbf{Z}^{\mathbf{i}}|\theta]e^{-\sum_j f^i(\mathbf{Z}^i{}_j)}\right\},$$

with

$$\int [\mathbf{Z}^{\mathbf{i}}|\theta]e^{-\sum_j f^i(\mathbf{Z}^i{}_j)}\,\mathrm{d}\mathbf{Z}^i \;= e^{-n_i\lambda(A_i)}\,.$$