On the Cosmetics of Exceedance Point Processes and Related Theory since 1983

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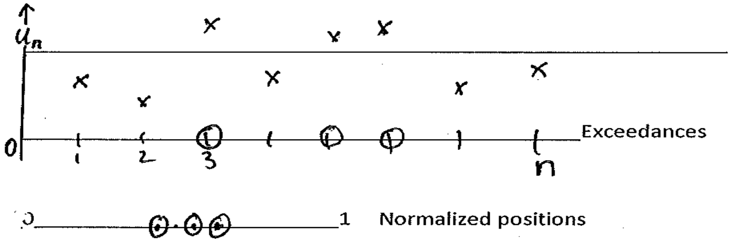
Exceedance Point Process(iid)

- If $\xi_1, \xi_2, \cdots, \xi_n$ are n iid r.v's with d.f F, the exceedances of a level u_n are the points i where $\xi_i > u_n$. The number S_n of these is binomial $S_n = B(n, p_n = 1 - F(u_n))$.
- If $n(1 - F(u_n)) \to \tau < \infty$, S_n converges in distribution to a Poisson r.v. with mean τ.
- To discuss the exceedances as a point process, we normalize occurrence times by n leading to exceedance points $i/n, 1 \leq i \leq n$ for which $\xi_i > u_n$, i.e. a point process on $(0, 1]$, $N_n = \{\frac{i}{n} : \xi_i > u_n, 1 \leq i \leq n\}$.
- $N_n(B) = \#\{\frac{i}{n} \in B : \xi_i > u_n\}$, Borel $B \subset (0, 1]$.

Exceedance Point Process (iid)



Exceedance Point Process (iid)

- If $I = (0, 1]$, then $N_n(I) = S_n \xrightarrow{d} N(I)$ for a Poisson Process N on $(0, 1]$ with intensity τ.

- For k disjoint intervals $I_1, I_2, \cdots, I_k \subset (0, 1]$, $\quad (N_n(I_1), N_n(I_2), \cdots, N_n(I_k))$ are independent and hence,

$$(N_n(I_1), N_n(I_2), \cdots, N_n(I_k)) \xrightarrow{d} (N(I_1), N(I_2), \cdots, N(I_k)).$$

Point Process Convergence

- Convergence of point processes (and random measures) in distribution is defined for much more abstract situations and requiring more than such multivariate convergence. But in this real line context, it follows from [Kallenberg, Theorem 4.2] that the general definition of convergence in distribution $N_n \stackrel{d}{\to} N$ is equivalent to
  $(N_n(I_1), N_n(I_2), \cdots, N_n(I_k)) \stackrel{d}{\to} (N(I_1), N(I_2), \cdots, N(I_k))$
  for disjoint intervals $I_j = (a_j, b_j] \subset (0, 1]$ provided N has no mass at any endpoint $a_j, b_j, N(a_j) = N(b_j) = 0, \text{a.s.} 1 \leq j \leq k$.
- This is the case for the Poisson Process N considered here, hence the exceedance point process $N_n \stackrel{d}{\to} N$, Poisson with intensity τ, when the ξ_i are iid and and $n(1 - F(u_n)) \stackrel{d}{\to} \tau$

Dependence - Strong Mixing

- Extension of CLT from iid to dependent sequences $\{\xi_n\}$ requires dependence restrictions such as strong mixing viz: $sup\{|P(A \cap B) - P(A)P(B)| : A \in \sigma(\xi_1, \cdots, \xi_j)$
  $B \in \sigma(\xi_{j+l}, \xi_{j+l+1}, \cdots), j = 1, 2, \cdots\} \to 0$, as $l \to \infty$.

- For EVT where maxima are more easily described statistically than sums one expects less restrictive dependence restrictions involving less events A,B than strong mixing.

Weaker Dependence Conditions($D, \Delta, \cdots$)

- Define a hierarchy of (array forms of) such less restrictive mixing conditions: For positive integers $i, j, i < j$, let M_{ij} be a class of sets $A \in \sigma(\xi_k : i \leq k \leq j)$. Write for positive integers $n, l < n$, $\alpha_{n,l} = sup\{|P(A \cap B) - P(A)P(B)| : A \in M_{1,j}, \quad B \in M_{j+l,n}, 1 \leq j \leq n - l\}$
- If $\{u_n\}$ is a sequence of real numbers, we say that the condition $\Delta(u_n)$ holds for the stationary process $\{\xi_n\}$ (relative to the family $\{M_{ij}\}$) if $\alpha_{n,l_n} \to 0$ as $n \to \infty$ for some sequence $l_n = o(n)$, or equivalently, $\alpha_{n,\epsilon n} \to 0$ for each $\epsilon > 0$.
- If M_{ij} is the class of sets of the form $\bigcap_r (\xi_{k_r} \leq u_n)$ for any choice of distinct $k_r, i \leq k_r \leq j$, this is the familiar $D(u_n)$ condition with $\alpha_{n,l} = max\{|F_{i_1,\cdots,i_p,j_1,\cdots,j_q}(u_n) - F_{i_1,\cdots,i_p}(u_n)F_{j_1,\cdots,j_q}(u_n)| : p + q \leq n - l, j_1 - i_p \geq l\}$ in terms of the indicated joint d.f.'s.

Equivalent Δ-Conditions

- $D(u_n)$ is an appropriate Δ condition to deal with asymptotic distribution of maxima. The corresponding M_{ij}'s are closed under intersections but not assumed to be σ -fields.
- For dealing with exceedance and related point processes, it will typically be assumed that the M_{ij} are (specific) σ-fields. In that case $\Delta(u_n)$ has the following equivalent definition.
- Write $\beta_{n,l} = sup\{|E(XY) - EXEY| : X \quad \text{is} \quad M_{1,j} - \text{meas}\}$. $Y \quad \text{is} \quad M_{j+l,n} - \text{meas}, \quad 0 \le X, Y \le 1$. Then $\Delta(u_n)$ holds iff $\beta_{n,l_n} \to 0$, some $l_n = o(n)$.
- In our applications, X, Y will be typically of the form $e^{-sN_n(I)}$ for point processes N_n, intervals I and $s > 0$.

Extremal Index for a stationary sequence

- Basic theorimo of classical EVT: $M_n = max(\xi_1, \xi_2, \cdots, \xi_n)$. If iid ξ_i have df F and $n(1 - F(u_n)) \to \tau \geq 0$, then $P(M_n \leq u_n) \to e^{-\tau}$ and conversely.
- For a stationary sequence satisfying $D(u_n)$ whenever $n(1 - F(u_n)) \to \tau, (u_n = u_n(\tau))$, if $P(M_n \leq u_n(\tau))$ converges for some τ, it converges for all τ and $limP(M_n \leq u_n(\tau)) = e^{-\theta\tau}$, for some $\theta, 0 \leq \theta \leq 1$. θ is the "Extremal index (EI)".
- $\theta = 1$ for iid sequences, $\theta = 0$ pathological.
- Here, we assume $\theta > 0$.

Exceedance Clustering in a stationary sequence

(Positive) dependence between successive ξ_i tends to encourage high positive values $\xi_i > u_n$ to be followed by another high value $\xi_{i+1} > u_n$ and hence a "cluster" of exceedances. Normalizing by n causes each cluster to shrink to a single position. For the exceedance point process N_n on $[0, 1]$, this suggests a limiting point process with multiple events.

Exceedance Clustering in a stationary sequence

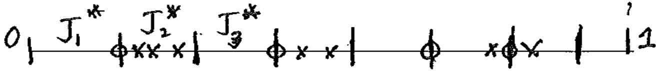
Assume $D(u_n)$ holds and E.I. $\theta > 0$.
Use a "block definition" of clusters:
let $k_n \to \infty$ satisfy $k_n(\alpha_{n,l_n} + l_n/n) \to 0$, and $r_n = [n/k_n]$. Let J_i be intervals $J_i = ((i-1)r_n + 1, (i-1)(r_n+2), \cdots, ir_n), 1 \leq i \leq k_n$ and $J_i^* = n^{-1}J_i$ which fill $(0,1]$ except for a partial interval $J_{k_n+1}^*$(ignored here).
Exceedances in a block $J_i(J_i^*)$ are regarded as a cluster located at the first point $((i-1)r_n + 1)/n$ of J_i^*. Cluster size=$N_n(J_i^*)$, cluster size distribution:

$$\begin{aligned}\pi_n(r) &= P\{N_n(J_1^*) = r | N_n(J_1^*) > 0\} \\ &= P\{N_n(J_1^*) = r | M_{r_n} > u_n\}, \quad r \geq 1\end{aligned}$$

Point Processes Considered

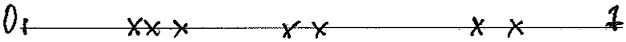
(i) Locations of exceedance clusters.
(ii) Exceedance point process.
(iii) Point process of excesses above u_n.
(iv) Point process of peak cluster values over threshold.

Point Process Considered



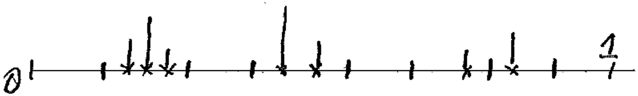
Normalized exceedance positions: x, cluster locations: o

Point Process Considered



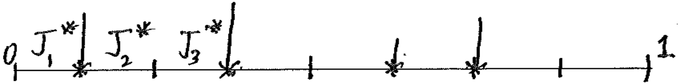
Exceedance point process

Point Process Considered



Point process of excess values.

Point Process Considered



Point process of cluster peaks

Assumptions to be made

Assume:

- $\{\xi_n\}$ stationary.
- Satisfies $D(u_n)$ where $u_n = u_n(\tau), (n(1 - F(u_n)) \to \tau)$, or some specified stronger Δ.
- $\{\xi_n\}$ has E.I. $\theta > 0$ so that $P(M_n \leq u_n) \to e^{-\theta\tau}$ (can be replaced by $P(M_n \leq u_n) \to e^{-\nu}, 0 < \nu < \infty$).

(i) Point Process N_n^* of cluster locations

N_n^* has points in [0,1] at normalized cluster locations $((i-1)r_n+1)/n$ if $J_i = \{(i-1)r_n+1, \cdots, ir_n\}$ contains an exceedance, $\xi_k > u_n$ for some $k \in J_i$.

Result:
Under $D(u_n)$, $N_n^* \stackrel{d}{\to} N$ a Poisson Process $P(\theta\tau)$ with intensity $\theta\tau$ on [0,1].
Easiest to prove from sufficient conditions of Kallenberg for simple point process limits, by showing:

(a) $EN_n^*(c,d] \to \theta\tau(d-c) = EN(c,d], 0 < c < d \leq 1$

(b) $P\{N_n(I) = 0\} \to P\{N(I) = 0\}$ when $I = \bigcup_1^k I_j$ for k disjoint subintervals $I_j = (c_j, d_j]$ of $(0,1]$.

Point Process N_n^* of cluster locations

(a) $P(M_n \leq u_n) - P^{k_n}\{M_{r_n} \leq u_n\} \to 0$, $P\{M_n \leq u_n\} \to e^{-\theta\tau}$and hence $P\{M_{r_n} > u_n\} \sim \theta\tau/k_n$ giving $EN^*(c,d) \sim (d-c)k_nP(M_{r_n} > u_n) \to \theta\tau(d-c) = EN(c,d]$.

(b) Consider $k = 2, \quad I = I_1 \cup I_2$. If I_1 and I_2 are separated by some $\epsilon > 0$, $D(u_n)$ gives

$$
\begin{aligned}
&|P(N_n^*(I) = 0) - P(N_n^*(I_1) = 0)P(N_n^*(I_2) = 0)| \\
&= |P(M(nI) \leq u_n) - P(M(nI_1) \leq u_n)P(M(nI_2) \leq u_n)| \\
&\leq \alpha_{n,\epsilon n} \to 0
\end{aligned}
$$

giving $P\{N_n^*(I) = 0\} \to e^{-\theta\tau(d_1-c_1)}e^{-\theta\tau(d_2-c_2)} = P\{N(I) = 0\}$
If I_1, I_2 abut, e.g. $c_2 = d_1$, shorten I_2 to $(c_2+\epsilon, d_2]$ and approximate. Note e.g. that
$0 \leq P(N_n^*(c+\epsilon, d] = 0) - P(N_n^*(c,d] = 0) \leq EN_n^*\{(c, c+\epsilon]\} \leq n(1-F(u_n))\epsilon \leq K\epsilon$, $K < \infty$.

(ii)Exceedance Point process N_n

$N_n =$ points at $\frac{i}{n} \in (0, 1]$ with $\xi_i > u_n$. Recall cluster size distribution π_n, $\pi_n(r) = P\{N_n(J_1^*) = r | M_{r_n} > u_n|, r = 1, 2, \cdots, \}$.

Result:

Assume Δ holds with $M_{j,k}(u_n) = \sigma\{\xi_s \leq u_n, \quad j \leq s \leq k\}$ and that the cluster size distribution converges $\pi_n \xrightarrow{d} \pi$ on 1,2,...
Then $N_n \xrightarrow{d} N$ where N is a Compound Poisson Process. $CP(\theta\tau, \pi)$ based on a Poisson Process with intensity $\theta\tau$ but having iid integer valued multiplicities with distribution π.

Exceedance Point process N_n

Method of Proof:

Extends previous method by showing:

(a) $N_n(I) \stackrel{d}{\to} N(I)$ each interval $I = (c, d] \subset (0, 1]$

(b) $N_n(I_1), \cdots, N_n(I_k)$ are asymptotically independent for each set of disjoint sub-intervals of (0,1].

(c) Hence $(N_n(I_1), \cdots, N_n(I_k)) \stackrel{d}{\to} (N(I_1,), \cdots, N(I_k))$ from which it follows that $N_d \stackrel{d}{\to} N$ (Kallenberg Theorem 4.2).

Exceedance Point process N_n

- Define the Laplace Transform of a non-negative r.v. ξ as $L_\xi(s) = Ee^{-s\xi}$,(more convenient here than characteristic functions). Non-neg r.v.'s $\xi_n \xrightarrow{d} \xi$ iff $L_{\xi_n}(s) \to L_\xi(s), s > 0$.

  (a): show that $Ee^{-sN_n(c,d]} \to Ee^{-sN(c,d]}, s > 0$. Just as $P(M_n \leq u_n) - P^{k_n}(M_{r_n} \leq u_n) \to 0$ by D it follows by the alternative form of Δ that $Ee^{-sN(0,1]} - E^{k_n}e^{-sN_n(J_1^*)} \to 0$.

Exceedance Point process N_n

- Since $N_n(J_1^*)$ has conditional distribution π_n given $N_n(J_1^*) > 0$,

$$Ee^{-sN_n(J_1^*)} = P(M_{r_n} \leq u_n) + P(M_{r_n} > u_n) \sum_1^\infty e^{-sr} \pi_n(r)$$
$$= 1 - \frac{\theta\tau}{k_n}(1 + o(1))(1 - \psi_n(s))$$

  where $\psi_n(s)$ is the L.T. of the distribution π_n, $\psi_n(s) \to \psi(s) = \sum_r e^{-sr}\pi_r$, L.T. of π, since $\pi_n \xrightarrow{d} \pi$.
- This all gives $Ee^{-sN_n(0,1]} \to e^{-\theta\tau(1-\psi(s))}$ the L.T. of a $CP(\theta\tau, \pi)$ r.v. so that $N_n(0,1] \xrightarrow{d} N(0,1]$. Similarly, $N_n(I) \xrightarrow{d} N(I)$ for each interval $I \subset (0,1]$ showing (a).

Exceedance Point process N_n

- (b): consider $k = 2$ and show asymptotic independence of $N_n(I_1), N_n(I_2)$ in the form

$$Ee^{-s_1 N_n(I_1) - s_2 N_n(I_2)} - Ee^{-s_1 N_n(I_1)} Ee^{-s_2 N_n(I_2)} \to 0$$

  directly from the alternative form of Δ, with r.v.'s $X = e^{-s_1 N_n(I_1)}$, $Y = e^{-s_2 N_n(I_2)}$ when I_1 and I_2 are separated by some $\epsilon > 0$, shortening say I_2 by ϵ when they abut and letting $\epsilon \to 0$. This extends to k disjoint intervals to give (b) and (c) to complete the proof.

(iii) Point Process P_n of excesses above u_n

P_n is defined to consist of the normalized exceedance points i/n for which $\xi > u_n$, but marked with the excess value $(X_i - u_n)$. We refer to P_n as a point process even though the multiplicities $(X_i - u_n)$ of events need not be integers. Reflecting this, convergence may require a normalization of the value of P_n and we consider convergence $a_n P_n \to N$ for some $a_n > 0$.

Define the "cluster mass distribution"

$$\pi_n(x) = P\{a_n P_n(J_1^*) \leq x | P_n(J_1^*) > 0\} = P\{a_n P_n(J_1^*) \leq x | M_{r_n} > u_n\}$$

now a distribution on $(0, \infty)$, not just 1,2,3....

Point Process P_n of excesses above u_n

Result(Assume E.I.)

- Let $\{\xi_n\}$ satisfy $\Delta(u_n)$ based on $M_{jk}(u_n) = \sigma\{(\xi_s - u_n)_+ : j \leq s \leq k\}$ where $(n(1 - F(u_n))) \to \tau$. Assume the cluster mass distribution π_n of $a_n P_n$ converges for some a_n to a non-degenerate limit π. Then $a_n P_n \stackrel{d}{\to} N, CP(\theta\tau, \pi)$.
- This may be shown in a similar way to the previous result for N_n, using $Ee^{-a_n s P_n(I)} - E^{k_n} e^{-a_n s P_n(J_1^*)} \to 0$ from Δ to give $a_n P_n(I) \stackrel{d}{\to} N(I)$ for an interval I and asymptotic independence of $P_n(I_1), \cdots, P_n(I_k)$ (from Δ) for disjoint $I_1, \cdots, I_k$.
- The event multiplicities in P_n are the limiting sums of excesses $(\xi_i - u_n)_+$ in a cluster (normalized by a_n). This contrasts with our final result involving cluster peaks.

(iv) The point process P_n^* of cluster peaks over threshold (POT)

- P_n^* consists of points at cluster locations, $\frac{((i-1)r_n+1)}{n}$ if $M(J_i) > u_n$ and carries mass multiplicity $\max_{j \in J_i}(\xi_i - u_n)_+$ generalizing POT for iid sequences with single point clusters. As with the iid case this applies when the d.f. F belongs to the domain of some EV distribution G, necessarily of the form $G(x) = e^{-\overline{H}(x)}$, $\overline{H}(x) = 1 - H(x)$, where $\frac{1-F(u+xg(u))}{1-F(u)} \to \overline{H}(x)$ as $u \to x_F$, some function $g(u) > 0$.

- The appropriate Δ condition will again be based on the σ-fields $M_{j,k} = \sigma\{(\xi_s - u_n)_+ : j \leq s \leq k\}$

The point process P_n^* of cluster peaks over threshold (POT)

Under these conditions the following holds for $u_n = u_n(\tau)$.
Lemma:

(i) $P\{a_n(M_{r_n} - u_n) > x\} \sim \frac{\theta\tau}{k_n}\overline{H}(x), \quad x > 0, \quad a_n = 1/g(u_n).$

(ii) $P\{a_n(M_{r_n} - u_n) \leq x | M_{r_n} > u_n\} \to H(x).$

(iii)

$$E^{k_n} exp\{-sa_n(M_{r_n} - u_n)_+\} \to exp\{-\theta\tau(1 - \psi(s))\}$$

$$\psi(s) = \int_0^\infty e^{-sx} dH(x)$$

This is provided by familiar means and leads to the following theorem:

The point process P_n^* of cluster peaks over threshold(POT)

Theorem: With the above conditions, $a_n P_n^* \xrightarrow{d} P^*$ where $a_n = 1/g(u_n)$ and P^* is $CP(\theta\tau, H)$. Further H has generalized Pareto form $1 - (1 + \xi x)_+^{-1/\xi}, \quad x > 0$

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