A local moment type estimator for the extreme value index in regression with random covariates

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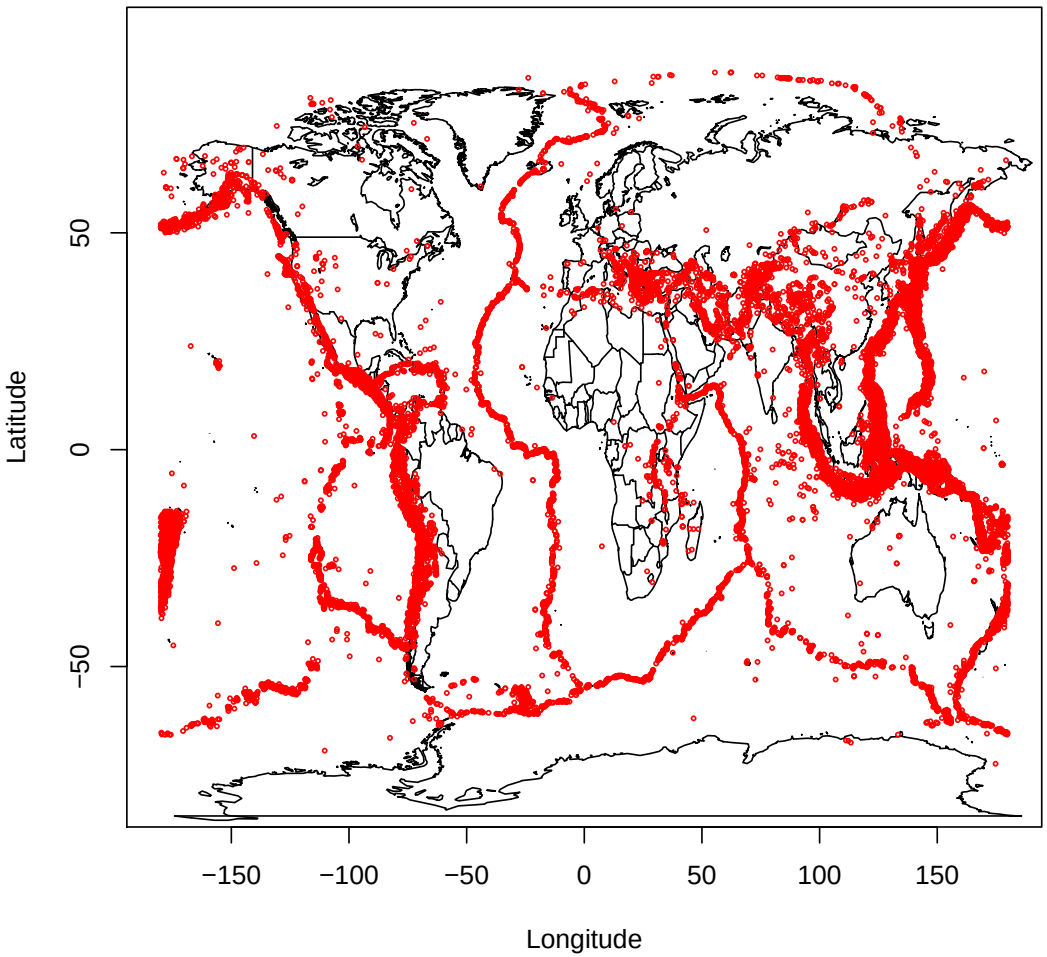
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Longitude

- Accurate modelling of the tail of the seismic moment distribution is important since large earthquakes cause heavy losses.
- The location of an earthquake is random, but we want to study the tail behaviour at a specific, fixed location.
- Previous attempts to model the tail behaviour, have relied on predefined zones where all data is assumed to be coming from the same distribution.

1. Motivating example
2. Estimation of $\gamma(x)$
3. Simulations
4. Dataanalysis

- Let (X_i, Y_i), $i = 1, \ldots, n$, be independent copies of the random vector $(X, Y) \in \mathbb{R}^p \times \mathbb{R}_+$.
- Assume X has a distribution with joint density g, and the conditional distribution function of Y given $X = x$ belongs to the max-domain of attraction of the generalized extreme value distribution, i.e. there exists a constant $\gamma(x)$ and a positive rate function $a(.; x)$, such that

$$\lim_{t \to \infty} \frac{U(ty; x) - U(t; x)}{a(t; x)} = D_{\gamma(x)}(y), \tag{1}$$

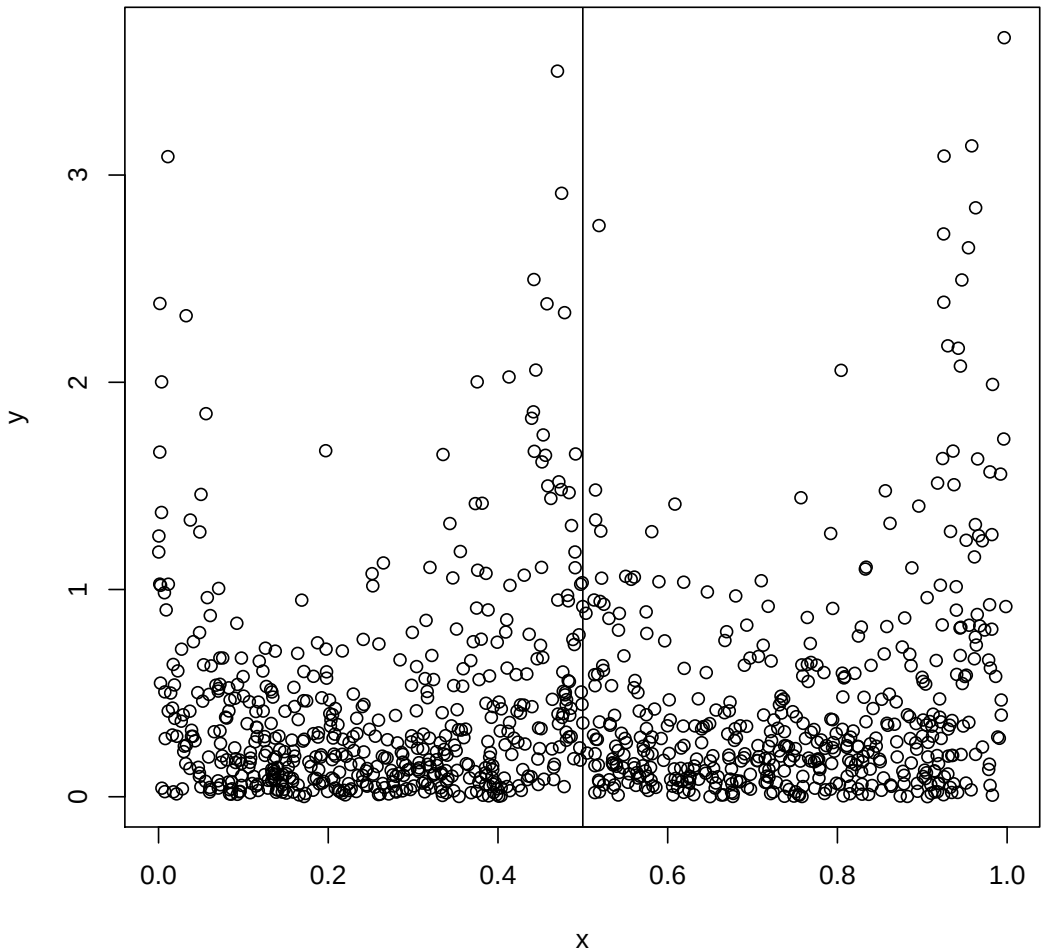
  with $D_{\gamma(x)}(y) := \int_1^y u^{\gamma(x)-1} du$ for every $y > 0$.
- The goal is to construct an estimator for $\gamma(x)$.

The basic building block for our estimator is

$$T_n^{(t)}(x, K) := \frac{1}{n} \sum_{i=1}^{n} K_{h_n}(x - X_i)(\ln Y_i - \ln \omega_n)_+^t \mathbf{1}\{Y_i > \omega_n\}$$

with $t = 0, 1, \ldots, 6$. We assume

- $K_{h_n}(x) := K(x/h_n)/h_n^p$, where K is a joint density on $\mathbb{R}^p$,
- h_n is a non-random sequence of bandwidths with $h_n \to 0$ if $n \to \infty$,
- ω_n is a local non-random threshold sequence satisfying $\omega_n \to y^*(x)$ if $n \to \infty$, where $y^*(x)$ is the right endpoint of $F(y; x)$.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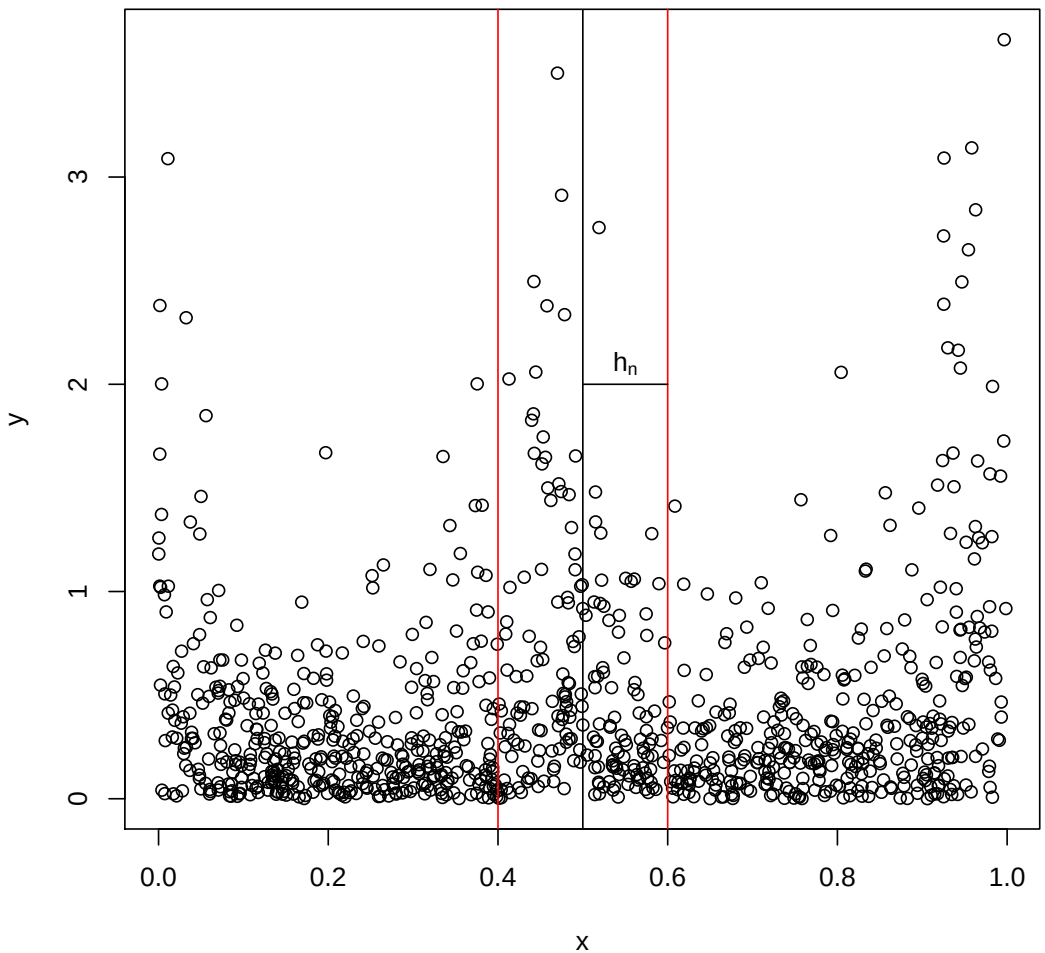
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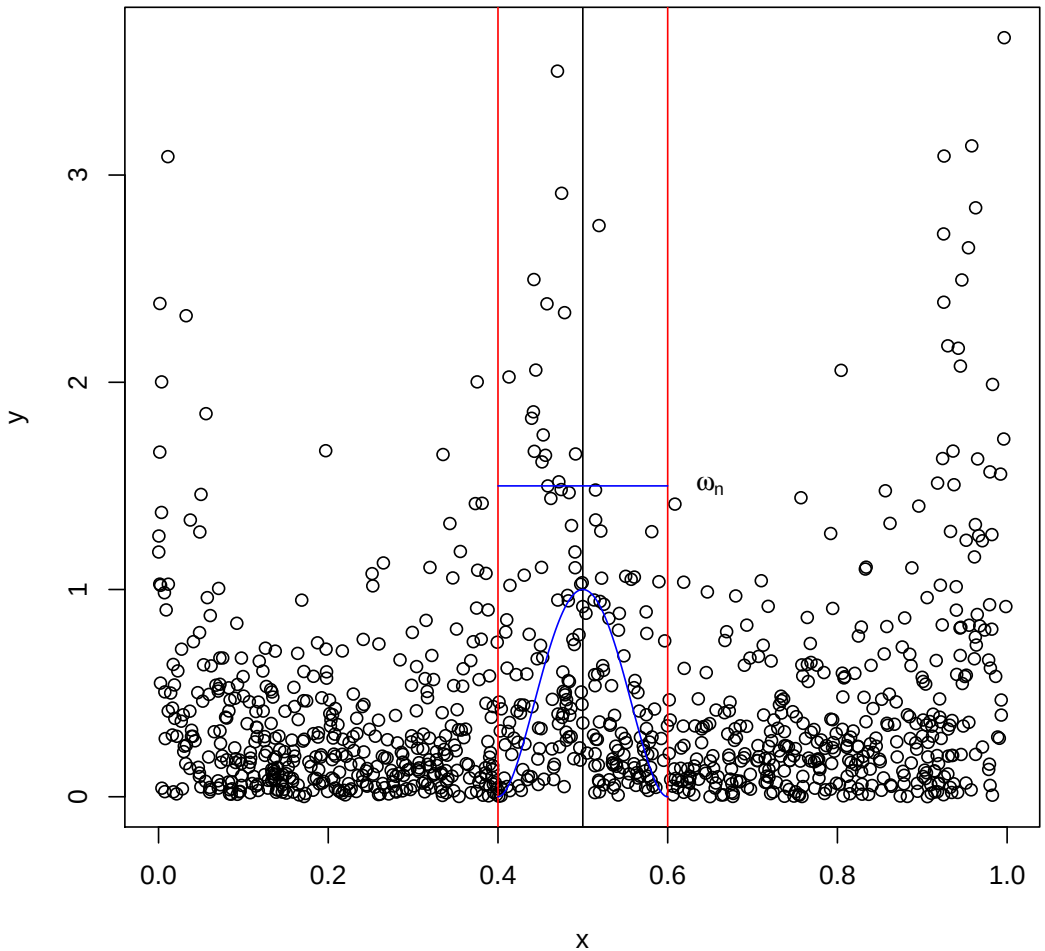
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ω_n

Assumptions

Assumption $(\mathcal{R})$ *There exists constants* $\gamma(x) \in \mathbb{R}$ *and* $\rho(x) \leq 0$, *a positive rate function* $a(.;x)$ *and a function* $A(.;x)$ *not changing sign ultimately, with* $A(t;x) \to 0$ *for* $t \to \infty$, *such that*

$$\lim_{t\to\infty} \frac{\frac{U(ty;x)-U(t;x)}{a(t;x)} - D_{\gamma(x)}(y)}{A(t;x)} = D_{\gamma(x),\rho(x)}(y),$$

with

$$D_{\gamma(x),\rho(x)}(y) := \int_1^y s^{\gamma(x)-1} \int_1^s u^{\rho(x)-1} du ds$$

for all $y > 0$.

Notation

Let

$$\overline{A}(t;x) := \frac{a(t;x)}{U(t;x)} - \gamma_+(x), \quad \gamma_+(x) := \max(0, \gamma(x))$$

and define

$$B(t;x) := \begin{cases} \overline{A}(t;x), & \gamma(x) > \rho(x) \\ A(t;x), & \gamma(x) < \rho(x) \end{cases}$$

and

$$a^*(t;x) := \begin{cases} a(t;x)\left(1 - \frac{B(t;x)}{\gamma(x)}\right), & \gamma(x) < \rho(x) = 0 \\ a(t;x)\left(1 - \frac{B(t;x)}{\rho(x)}\right), & \gamma(x) < \rho(x) < 0 \\ a(t;x)\left(1 + \frac{2B(t;x)}{\gamma(x)}\right), & \rho(x) < \gamma(x) < 0 \\ a(t;x), & \rho(x) < \gamma(x) = 0 \end{cases}.$$

Furthermore let

$$\ell(x) := \lim_{t\to\infty}\left(U(t;x) - \frac{a(t;x)}{\gamma(x)}\right) \in \mathbb{R} \quad \text{for } 0 < \gamma(x) < -\rho(x).$$

The local moment estimator

We consider the estimator

$$\widehat{\gamma}_n(x) := \frac{T_n^{(1)}(x, K_1)}{T_n^{(0)}(x, K_0)} + 1 - \frac{1}{2}\left(1 - \frac{\left(\frac{T_n^{(1)}(x,K_1)}{T_n^{(0)}(x,K_0)}\right)^2}{\frac{T_n^{(2)}(x,K_2)}{T_n^{(0)}(x,K_0)}}\right)^{-1}.$$

Define

$$m(\omega_n, t; x) := \mathbb{E}\left[(\ln Y - \ln \omega_n)_+^t \mathbf{1}\{Y > \omega_n\}; X = x\right].$$

Case (i), $t = 0$:

$$m(\omega_n, t; x) = \overline{F}(\omega_n; x).$$

Assume $(\mathcal{R})$ with $\rho(x) \neq \gamma(x)$, then for $\omega_n \to y^*(x)$ we have that:
Case (ii), $t = 1$:

$$m(\omega_n, t; x) = \begin{cases} \overline{F}(\omega_n; x) \left[\gamma(x) + \frac{1}{1-\widetilde{\rho}(x)} B\left(\frac{1}{\overline{F}(\omega_n;x)}; x\right)(1 + o(1))\right], & \gamma(x) > 0 \\ \overline{F}(\omega_n; x) \frac{a^*\left(\frac{1}{\overline{F}(\omega_n;x)};x\right)}{U\left(\frac{1}{\overline{F}(\omega_n;x)};x\right)} \left[\frac{1}{1-\gamma(x)} + b^{(1)}_{\gamma(x),\rho(x)} B\left(\frac{1}{\overline{F}(\omega_n;x)}; x\right)(1 + o(1))\right], & \gamma(x) \leq 0 \end{cases},$$

where

$$\widetilde{\rho}(x) := \begin{cases} \rho(x), & I(x) = 0 \text{ or } -\rho(x) < \gamma(x) \\ -\gamma(x), & \text{otherwise} \end{cases}$$

and

$$b^{(1)}_{\gamma(x),\rho(x)} := \begin{cases} \frac{1}{\gamma(x)(1-\gamma(x))^2}, & \gamma(x) < \rho(x) = 0 \\ \frac{1}{\rho(x)(1-\gamma(x)-\rho(x))}, & \gamma(x) < \rho(x) < 0 \\ -\frac{2-3\gamma(x)}{\gamma(x)(1-\gamma(x))(1-2\gamma(x))}, & \rho(x) < \gamma(x) < 0 \\ -1, & \rho(x) < \gamma(x) = 0 \end{cases}.$$

Case (iii), $t = 2$:

$$m(\omega_n, t; x) = \begin{cases} \overline{F}(\omega_n; x) \left[2\gamma^2(x) + \frac{2\gamma(x)(2-\widetilde{\rho}(x))}{(1-\widetilde{\rho}(x))^2} B\left(\frac{1}{\overline{F}(\omega_n;x)}; x \right) (1 + o(1)) \right], \\ \overline{F}(\omega_n; x) \left(\frac{a^*\left(\frac{1}{\overline{F}(\omega_n;x)}; x \right)}{U\left(\frac{1}{\overline{F}(\omega_n;x)}; x \right)} \right)^2 \left[\frac{2}{(1-\gamma(x))(1-2\gamma(x))} + b^{(2)}_{\gamma(x),\rho(x)} B\left(\frac{1}{\overline{F}(\omega_n;x)}; x \right) (1 + o(1)) \right], \end{cases}$$

where

$$b^{(2)}_{\gamma(x),\rho(x)} := \begin{cases} \frac{2(2-3\gamma(x))}{\gamma(x)(1-\gamma(x))^2(1-2\gamma(x))^2}, & \gamma(x) < \rho(x) = 0 \\ \frac{2(2-2\gamma(x)-\rho(x))}{\rho(x)(1-\gamma(x))(1-\gamma(x)-\rho(x))(1-2\gamma(x)-\rho(x))}, & \gamma(x) < \rho(x) < 0 \\ -\frac{8-18\gamma(x)}{\gamma(x)(1-\gamma(x))(1-2\gamma(x))(1-3\gamma(x))}, & \rho(x) < \gamma(x) < 0 \\ -6, & \rho(x) < \gamma(x) = 0 \end{cases}.$$

Further assumptions

Assumption $(\mathcal{G})$ *There exists* $c_g > 0$ *and* $\eta_g > 0$ *such that* $|g(x) - g(z)| \leq c_g \|x - z\|^{\eta_g}$ *for all* $x, z \in \mathbb{R}^p$.

Assumption $(\mathcal{F})$ *The conditional expectation*

$$m(\omega_n, t; x) = \mathbb{E}\left[(\ln Y - \ln \omega_n)_+^t \, \mathbf{1}\{Y > \omega_n\}; X = x\right]$$

satisfies that, for $\omega_n \to y^*(x)$, $h_n \to 0$,

$$\Phi(\omega_n, h_n; x) := \sup_{t \in \{0,1,\ldots,6\}} \sup_{z \in \Omega} \left| \frac{m(\omega_n, t; x - z h_n)}{m(\omega_n, t; x)} - 1 \right| \to 0 \text{ if } n \to \infty.$$

Assumption $(\mathcal{K})$ K *is a bounded density function on* $\mathbb{R}^p$, *with support* Ω *included in the unit hypersphere in* $\mathbb{R}^p$.

Asymptotic expansion of main statistic

Let

$$\widetilde{m}_n(K, t; x) := \mathbb{E}\left[T_n^{(t)}(x, K)\right].$$

Assume $(\mathcal{R})$ with $\rho(x) \neq \gamma(x)$, $(\mathcal{G})$, $(\mathcal{K})$ and $(\mathcal{F})$. For all $x \in \mathbb{R}^p$ where $g(x) > 0$ we have for $n \to \infty$ with $h_n \to 0$ and $\omega_n \to y^*(x)$, then

$$\widetilde{m}_n(K, t; x) = m(\omega_n, t; x)\, g(x) \left(1 + O(h_n^{\eta_g}) + O\left(\Phi\left(\omega_n, h_n; x\right)\right)\right).$$

Standardization

Let

$$\widetilde{T}_n^{(t)}(x,K) := \begin{cases} \frac{1}{\overline{F}(\omega_n;x)g(x)}\, T_n^{(t)}(x,K), & \gamma(x) > 0 \\ \left(\frac{U\left(\frac{1}{\overline{F}(\omega_n;x)};x\right)}{a^*\left(\frac{1}{\overline{F}(\omega_n;x)};x\right)}\right)^t \frac{1}{\overline{F}(\omega_n;x)g(x)}\, T_n^{(t)}(x,K), & \gamma(x) \leq 0 \end{cases}$$

and

$$\mathbb{T}'_n := \left[\widetilde{T}_n^{(0)}(x,K_0), \widetilde{T}_n^{(1)}(x,K_1), \widetilde{T}_n^{(2)}(x,K_2)\right].$$

Asymptotic normality

Assume $(\mathcal{R})$ with $\rho(x) \neq \gamma(x)$, $(\mathcal{G})$ and $(\mathcal{F})$ are satisfied and that the kernel functions K_0, K_1 and K_2 satisfy $(\mathcal{K})$. For all $x \in \mathbb{R}^p$ where $g(x) > 0$ we have that if $h_n \to 0$, $\omega_n \to y^*(x)$ and $nh_n^p \overline{F}(\omega_n; x) \to \infty$ for $n \to \infty$, then

$$\sqrt{nh_n^p \overline{F}(\omega_n; x)} \left[\mathbb{T}_n - \mathbb{E}(\mathbb{T}_n)\right] \xrightarrow{D} N_3(0, \Sigma),$$

where the elements of Σ are given by

$$\Sigma_{j,k} := \begin{cases} \frac{(j+k)!\|K_j K_k\|_1 \gamma^{j+k}(x)}{g(x)}, & \gamma(x) > 0 \\ \frac{(j+k)!\|K_j K_k\|_1}{g(x)\Pi_{i=0}^{j+k}(1-i\gamma(x))}, & \gamma(x) \leq 0 \end{cases}, \quad j,k = 0,1,2.$$

Asymptotic normality of estimator

If additionally, $\sqrt{nh_n^p \overline{F}(\omega_n; x)} B\left(\frac{1}{\overline{F}(\omega_n;x)}; x\right) \to \lambda(x)$ for some constant $\lambda(x) \in \mathbb{R}$, $nh_n^{p+2\eta_g} \overline{F}(\omega_n; x) \to 0$ and $nh_n^p \overline{F}(\omega_n; x) \Phi^2(\omega_n, h_n; x) \to 0$, then

$$\sqrt{nh_n^p \overline{F}(\omega_n; x)} (\widehat{\gamma}_n(x) - \gamma(x)) \xrightarrow{D} N\left(\lambda(x)\mu, V'\Sigma V\right),$$

where

$$\mu = \begin{cases} \frac{(1-\gamma(x))(1-2\gamma(x))}{(1-\gamma(x)-\rho(x))(1-2\gamma(x)-\rho(x))} & \text{if } \gamma(x) < \rho(x) \leq 0 \\ -\frac{\gamma(x)(1+\gamma(x))}{(1-\gamma(x))(1-3\gamma(x))} & \text{if } \rho(x) < \gamma(x) \leq 0 \\ \frac{\gamma(x)(1-\rho(x))+\rho(x)}{\gamma(x)(1-\rho(x))^2} & \text{if } 0 \leq -\rho(x) < \gamma(x) \text{ or } (0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) = 0) \\ \frac{\gamma(x)}{(1+\gamma(x))^2} & \text{if } \gamma(x) + \rho(x) = 0 \text{ or } (0 < \gamma(x) < -\rho(x) \text{ and } \ell(x) \neq 0) \end{cases},$$

and

$$V := \begin{bmatrix} 1 - \gamma(x) \\ 1 - \frac{2}{\gamma(x)} \\ \frac{1}{2\gamma^2(x)} \end{bmatrix}$$

when $\gamma(x) > 0$, while

$$V := \begin{bmatrix} (1 - \gamma(x))(1 - 2\gamma(x)) \\ -2(1 - \gamma(x))^2(1 - 2\gamma(x)) \\ \frac{1}{2}(1 - \gamma(x))^2(1 - 2\gamma(x))^2 \end{bmatrix}$$

when $\gamma(x) \leq 0$.

Datadriven approach

- Datadriven approach: First select bandwidth

$$h_{cv} := \underset{h_n \in \mathcal{H}}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\mathbf{1}\{Y_i \leq Y_j\} - \hat{F}_{n,-i}(Y_j|X_i) \right)^2, \quad (2)$$

where $\mathcal{H}$ is a grid of values for h_n and

$$\hat{F}_{n,-i}(y|x) := \frac{\sum_{j=1, j\neq i}^{n} K_{h_n}(x - X_j)\mathbf{1}\{Y_j \leq y\}}{\sum_{j=1, j\neq i}^{n} K_{h_n}(x - X_j)}.$$

- Next, compute the estimates of $\gamma(x)$ for $k = 5, 6, , \ldots, \frac{n^*}{2}$, where n^* is the number of observations in the window.
- Split the estimates into blocks of size $\sqrt{n^*/2}$, and choose the median estimate in the block with the smallest standard deviation.

Reversed Burr$(\eta(x), \tau(x), \lambda(x))$

The *reversed Burr*$(\eta(x), \tau(x), \lambda(x))$ distribution, left-truncated at 0 and with right endpoint $y^*(x)$,

$$F(y;x) = 1 - \left(\frac{\eta(x) + y^*(x)^{-\tau}}{\eta(x) + (y^*(x) - y)^{-\tau(x)}} \right)^{\lambda(x)}, \quad 0 < y < y^*(x); \lambda(x), \eta(x), \tau(x) > 0.$$

for which $\gamma(x) = -\frac{1}{\lambda(x)\tau(x)}$ and $\rho(x) = -\frac{1}{\lambda(x)}$. We use $\eta(x) = 3$, $y^*(x) = 5$ and

$$\gamma(x) = -\frac{1}{2}\left(\frac{1}{10} + \sin(\pi x)\right)\left(\frac{11}{10} - \frac{1}{2}\exp\left(-64\left(x - \frac{1}{2}\right)^2\right)\right),$$

$\lambda(x)$	$\hat{\gamma}_n(x)$		$\tilde{\gamma}_4^{RP,1}(x)$	
	Bias	MSE	Bias	MSE
0.5	**0.1531** (0.0017)	**0.0398** (0.0009)	0.2168 (0.0026)	0.0831 (0.0022)
1	**0.2416** (0.0024)	**0.0916** (0.0017)	0.3105 (0.0026)	0.1388 (0.0025)
1.5	**0.3582** (0.0029)	**0.1843** (0.0027)	0.4130 (0.0029)	0.2154 (0.0031)
2	**0.4039** (0.0032)	**0.2168** (0.0031)	0.4843 (0.0033)	0.2840 (0.0042)

Table: Performance of $\hat{\gamma}_n(x)$ and $\tilde{\gamma}_4^{RP,1}(x)$. The results are averaged over $N = 500$ simulations with $n = 1000$ observations.

$Burr(\eta(x), \tau(x), \lambda(x))$

The $Burr(\eta(x), \tau(x), \lambda(x))$ distribution,

$$1 - F(y; x) = \left(\frac{\eta(x)}{\eta(x) + y^{\tau(x)}} \right)^{\lambda(x)}, \quad y > 0; \eta(x), \tau(x), \lambda(x) > 0,$$

for which $\gamma(x) = \frac{1}{\lambda(x)\tau(x)}$ and $\rho(x) = -\frac{1}{\lambda(x)}$. We consider the case

$$\gamma(x) = \frac{1}{2}\left(\frac{1}{10} + \sin(\pi x)\right)\left(\frac{11}{10} - \frac{1}{2}\exp\left(-64\left(x - \frac{1}{2}\right)^2\right)\right),$$

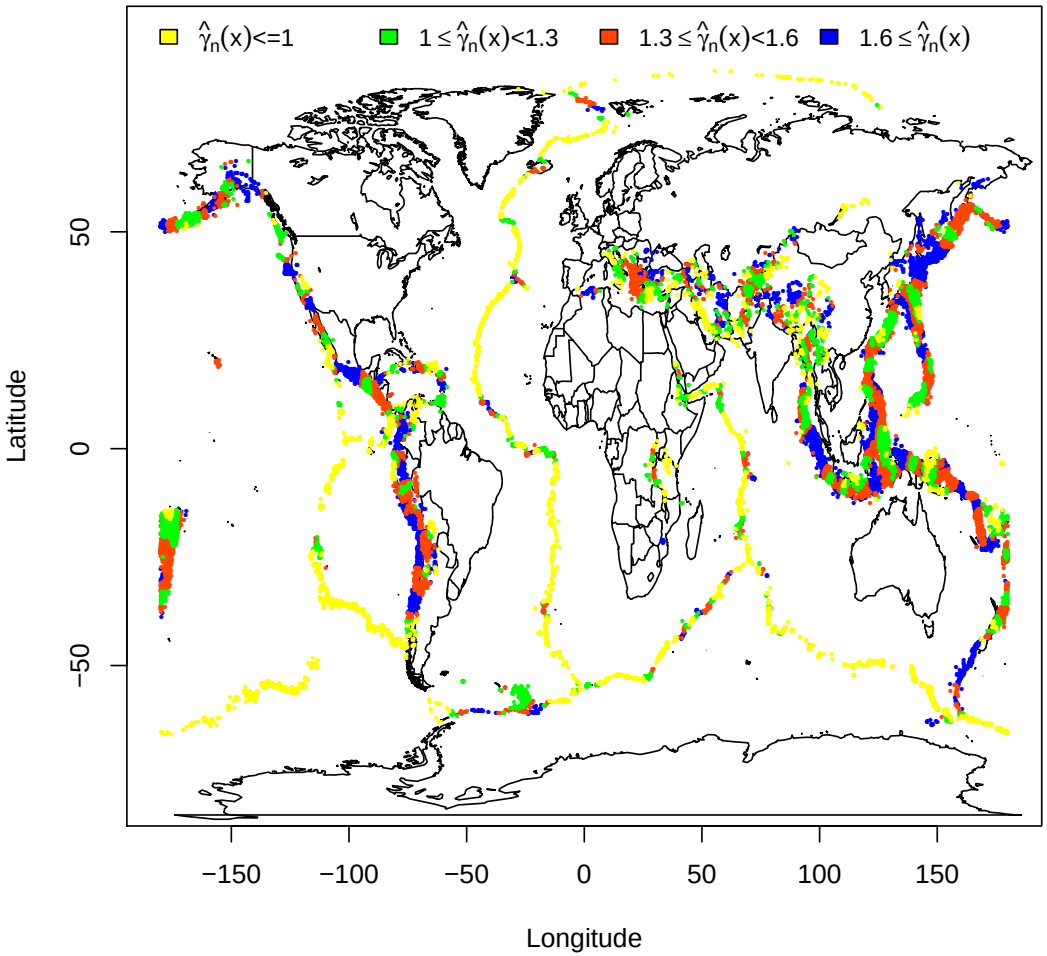
with $\eta(x) = 1$ and different but fixed values of $\lambda(x)$.

$\lambda(x)$	$\hat{\gamma}_n(x)$		$\tilde{\gamma}_4^{RP,1}(x)$	
	Bias	MSE	Bias	MSE
0.5	**0.1106** (0.0015)	**0.0200** (0.0006)	0.2262 (0.0031)	0.0849 (0.0026)
1	**0.1128** (0.0016)	**0.0208** (0.0006)	0.2310 (0.0032)	0.0856 (0.0027)
1.5	**0.1206** (0.0016)	**0.0234** (0.0006)	0.2397 (0.0033)	0.0905 (0.0025)
2	**0.1413** (0.0017)	**0.0315** (0.0007)	0.2513 (0.0032)	0.1026 (0.0027)

Table: Performance of $\hat{\gamma}_n(x)$ and $\tilde{\gamma}_4^{RP,1}(x)$. The results are averaged over $N = 500$ simulations with $n = 1000$ observations.

Earthquake data

- Earthquakedata is from the Global Centroid Moment Tensor database.
- Contains information about, among others, longitude, latitude and seismic moment of earthquakes that have occurred between 1976 and present (33872 earthquakes).
- The variable of main interest is the Seismic moment (measured in dyne-centimeters), and as covariate, we use the distance to every other earthquake.
- Estimate $\gamma(x)$ at all locations where an earthquake has appeared.


$\hat{\gamma}_n(x)<=1$
$1 \le \hat{\gamma}_n(x)<1.3$
$1.3 \le \hat{\gamma}_n(x)<1.6$
$1.6 \le \hat{\gamma}_n(x)$
Latitude
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Longitude