Penultimate Approximations: Past, Present . . . and Future?

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(joint work with Paula Reis, Luísa Canto e Castro & Sandra Dias)

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Repeating what I said in Shanghai two months ago, and last Sunday after the Dinner, it was indeed true that when Richard Davis at EVA 2009 spoke about VIMEIRO’s meeting as EVA–0, and when read at EVA 2013 website:

“It has been 30 years since the so-called zero-th EVA conference took place in 1983 in Vimeiro, a small town near the beach in Portugal” . . .

I indeed felt

Some “Nostalgia” . . .

And now, here we are in VIMEIRO, essentially thanks to my dear colleagues and friends, Antónia, Isabel and Manela.
But possibly it will be necessary to renumber **EVA-0**

Prior to **EVA** meetings, I think we cannot forget:


But let’s go back to **VIMEIRO 1983**:
This was the unique picture I had from Vimeiro's meeting—1983
But the one at EVA 2013 website is better—and with Feridun
Some of the invited speakers:

Clive Anderson  
Paul Deheuvels  
Benjamim Epstein  
Janos Galambos  
Laurens de Haan  
Ross Leadbetter  
Georg Lindgren  
Yashaswini Mittal*  
James Pickands III  
Sid Resnick  
Holger Rootzen  
Masaaki Sibuya  
J Tiago de Oliveira  
Jef Teugels  
Ishay Weissman

Other speakers:

Barry Arnold  
Richard Davis  
Anthony Davison  
Ivette Gomes*  
Abraham Hasofer  
David Mejzler  
Juerg Huesler  
Rolf-Dieter Reiss  
Dan Rosbjerg  
Richard Smith  
Feridun Turkman  
Jose Villasenor

Portuguese PhD students:

Teresa Alpuim*  
Emília Athayde*  
Isabel Barão*  
Eugénia Graça Martins*  
Helena Iglésias Pereira*  
Fátima Miguens*  
Manuela Neves*  
Fernando Rosado
ORGANIZERS

TIAGO  IVETTE  FERIDUN

SHADOW ORGANIZERS

ANTÓNIA  DINIS
And what about EVT 2013?

**Vimeiro 1983 GANG (18)**

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<th>GOMES, Ivette</th>
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<tr>
<td>ANDERSON, Clive</td>
<td>HUESLER, Juerg</td>
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<td>BARÃO, Isabel</td>
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**Senior intervenients (23)**

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<td>BRITO, Margarida</td>
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<td>CAROLINO, Elisabete</td>
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<td>CANTO E CASTRO, Luísa</td>
<td>GIRARD, Stephane</td>
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<td>CHAVEZ-DEMOULIN, Valérie</td>
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<td>KRATZ, Marie</td>
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<td>FERREIRA, Ana</td>
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**Young intervenients (28)**

<table>
<thead>
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<th>MARTINS, Ana Paula</th>
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<td>CAEIRO, Frederico</td>
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<td>DEMATTEO, Antoine</td>
<td>NASCIMENTO, Fernando</td>
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<td>DIAS, Alexandra</td>
<td>NORTHROP, Paul</td>
<td>SEIFERT, Miriam Isabel</td>
</tr>
<tr>
<td>DIAS, Sandra</td>
<td>NUNES, Sandra</td>
<td>SERRA MOCHALES, Isabel</td>
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<tr>
<td>FERREIRA, Marta</td>
<td>OSMANN, Michael</td>
<td>SHABY, Benjamin</td>
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<td>PADILLA, Maria</td>
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<td>FREITAS, Jorge</td>
<td>PADOAN, Simone</td>
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<td>GOUVEIA REIS, Délia</td>
<td>PENALVA, Helena</td>
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<td>KIRILIOUK, Anna</td>
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It was indeed very gratifying to see so many young people.
This means that the field is alive and promising . . .
But, let’s go into Science . . .
And like I did at EVA’13, I dedicate this talk to the memory of TIAGO de OLIVEIRA (1928–1992)...

Chair of the OC of the **NATO ASI on SEA**, and one of the main responsible for my change from **ALGEBRA** to **PROBABILITY** and **STATISTICS** ...
WHY PENULTIMATE APPROXIMATIONS AGAIN?

HISTORICAL FACTS:

- Clive Anderson, my Ph.D. supervisor, sequentially provided me with different topics of research, among which *rates of convergence and penultimate approximations* . . . , one of his favourite topics, I think . . .

- Among the articles I read during my stay in Sheffield, UK (1975–1978), the one that influenced me most was possibly the one by Fisher & Tippett (1928), and the aforementioned topic.

- And indeed I still think there is some kind of *magic* in this topic, because this my first passion has been intermittently revisited after my Ph.D. thesis [Gomes, 1978]. I’m thinking on Gomes (1982a; 1982b; 1984; 1986; 1994), Gomes & Pestana (1984) and Gomes & de Haan (1999).
AND NOW, THE PRESENT...and the FUTURE?

- Recently, my colleague and friend, Luísa Canto e Castro, one of my first Ph.D. students, now General Director of the Dept. of Statistics of the Ministry of Education and Science, has supervised Paula Reis, who got her Ph.D. degree in 2012, in a topic relating penultimate approximations and reliability of large and coherent systems.

- And just by chance, but enthusiastically, I was involved in this research...and in a way that I have really appreciated, because I was able to find a possible strong link between this topic in the field of extreme value theory (EVT) and another field I love, statistical process control and reliability.

- And sincerely, I think that there is still a lot to be done...provided we have time and patience for that.
OUTLINE

• **Importance of Order Statistics (OS) in Reliability** — In reliability theory any coherent system can be represented as either a series-parallel (SP) or a parallel-series (PS) system, and its lifetime can thus be written as the *minimum of maxima* or the *maximum of minima*.

• **Main Limiting Results in Extreme Value Theory (EVT)** — For large-scale coherent systems it is sensible to assume that the number of system components goes to infinity. Then, the possible non-degenerate extreme value distributions (EVD) either for maxima (EV$_{MD}$) or for minima (EV$_{mD}$) are eligible candidates for the system reliability or at least for the finding of adequate lower and upper bounds for such a reliability.
• **Ultimate and Penultimate Models for the Sequence of RFs of a Regular Homogeneous PS System** — The identification of the possible ultimate limit laws for the system reliability of homogeneous PS systems is sketched. In most situations such non-degenerate limit laws are better approximated by an adequate penultimate distribution.

• **A Small-Scale Simulation Study** — Dealing with regular and homogeneous PS systems, we assess the gain in accuracy when a penultimate approximation is used instead of a ultimate limiting approximation.
FUTURE ... and out of the scope of this presentation

- Similar results can be straightforwardly obtained for homogeneous SP systems.

- **Generalizations** of the classical i.i.d. set-up are feasible and in progress.

- More intricate but similar work can be done for non-homogeneous systems.

- **Applications to real data** are also in progress.
Introduction

- **Motivation:** The study of the exact reliability function (RF) of complex technological systems with lifetime $T$, given by

$$R_T(t) := \mathbb{P}(T > t) = 1 - F_T(t),$$

can be an intractable problem due to:

- their large number of components;
- the way the operating process uses such components.

- **Examples:** Among others, we mention transport networks of oil, gas and water, telecommunication and electrical energy distribution networks, charge and discharge networks.
Our strategy is essentially the following:

1. Assuming that the number of components of a system $S$ goes to infinity,

   **asymptotic EV or ultimate models**
   \[
   \Downarrow
   \]
   often provide a **good interpretation** of the **RF** of $S$

2. Considering a **fixed large number** of components,

   **pre-asymptotic or penultimate models**
   \[
   \Downarrow
   \]
   provide an **improvement of the convergence rate**
   \[
   \Downarrow
   \]
   and a **better approximation** to the **RF** of $S$. 
1. Importance of OS in Reliability

- Let $T$ denote the lifetime of a coherent structure with $n$ components, with lifetimes $(T_1, \ldots, T_n)$, possibly with the same common d.f. $F$.
- Let us denote $(T_{1:n} \leq \cdots \leq T_{n:n})$ the sample of associated ascending OS, with $T_{1:n} = \min_{1 \leq i \leq n} T_i$, $T_{n:n} = \max_{1 \leq i \leq n} T_i$.
- The main importance of OS in reliability lies on the fact that the r.v. $T$ can always be written as a function of OS associated to the r.v.’s $T_i$, $1 \leq i \leq n$. See GNEDENKO, BELAYEV & SOLOVYEV, 1969, Mathematical Methods of Reliability Theory, among others.
- Indeed, we have $T = T_{I:n}$, where $I$ is a discrete r.v. with support $\{1, 2, \ldots, n\}$. The vector $s := (s_1, s_2, \ldots, s_n)$, with $s_i := \mathbb{P}(I = i)$, $1 \leq i \leq n$, is the so-called signature of the system [SAMANIEGO, 1985, IEEE Trans. Reliab.].
Moreover, we state the following relevant result in reliability:

**Theorem 1** (BARLOW & PROSCHAN, 1975, *Statistical Theory of Reliability and Life Testing: Probability Models*). Any coherent structure can be represented either as a SP (a series structure with components connected in parallel) or a PS (a parallel structure with components connected in series) structure.

**Example 1.** Let us consider the simple bridge structure:
• In order to find the aforementioned representations in Theorem 1 we need to identify the so-called
  – minimal paths — paths without irrelevant components that enable the operation of the system, and the so-called
  – minimal cuts — a set of relevant components that will imply the failure of the system whenever removed.

• In the bridge structure, we have:
  – Minimal paths: \{1,4\}, \{2,5\}, \{1,3,5\}, \{2,3,4\};
  – Minimal cuts: \{1,2\}, \{4,5\}, \{1,3,5\}, \{2,3,4\}.

• Consequently, we have the following representations of the initial bridge system:
and
We can thus write,

\[ T = \max (\min(T_1, T_3, T_5), \min(T_2, T_3, T_4), \min(T_1, T_4), \min(T_2, T_5)) \]

as well as

\[ T = \min (\max(T_1, T_2), \max(T_4, T_5), \max(T_1, T_3, T_5), \max(T_2, T_3, T_4)). \]

- We obviously need to pay attention to the strong dependence of the different r.v.'s either in the overall \textit{max} or \textit{min} operators.

- But we can build \textit{lower} and \textit{upper bounds} for the reliability on the basis of the \textit{minimal cuts} (assuming they are independent) and \textit{minimal paths} (assuming they are disjoint), respectively.
Generally speaking, let \( P_j, 1 \leq j \leq p = p_n \), denote the minimal paths, and \( C_j, 1 \leq j \leq s = s_n \), the minimal cuts. Then, and for non-necessarily i.d. components, we have

\[
\prod_{j=1}^{s} \left( 1 - \prod_{i \in C_j} (F_i(t)) \right) \leq R_T(t) \leq 1 - \prod_{j=1}^{p} \left( 1 - \prod_{i \in P_j} (1 - F_i(t)) \right).
\]

For sake of simplicity, we now assume that all minimal paths have the same size \( l = l_n \) and that all minimal cuts have a size \( r = r_n \) (the so-called regular system), and that \( R_i(t) = R(t) \), \( 1 \leq i \leq n \) (the so-called homogeneous system). Then, with
\( L_{SP}(t) := (1 - (1 - R(t))^{r_n})^{s_n} =: (1 - M_{rn}(t))^{s_n} \)

and

\( U_{PS}(t) := 1 - (1 - R^{l_n}(t))^{p_n} =: 1 - (m_{ln}(t))^{p_n} \),

we get

\[ L_{SP}(t) \leq R_T(t) \leq U_{PS}(t), \]

\[ n = r_n s_n = l_n p_n. \]

The above lower and upper bounds are quite reliable, particularly when compared with the crude lower and upper bounds respectively given by associated series and parallel systems, with all the \( n \) components of our system \( S \), given by

\[ L_S(t) := 1 - m_n(t) = (1 - F(t))^n = R^n(t) \]
and

\[ U_P(t) := 1 - M_n(t) = 1 - F^n(t) = 1 - (1 - R(t))^n. \]

We obviously get

\[ L_S(t) \leq R_T(t) \leq U_P(t), \]

but \( L_{SP} \) and \( L_{PS} \) are much more accurate, as can be seen from the following Figure, where we consider the static counterpart of the RF, writing \( p := R(t) \), and represent for \( l_n = r_n = 4 \) and \( s_n = p_n = 15 \) (i.e. \( n = 60 \)), the lower bounds

\[ L_S = p^n \quad \text{and} \quad L_{SP} = \left( 1 - (1 - p)^r_n \right)^s_n, \]

as well as the upper bounds

\[ U_P := 1 - (1 - p)^n \quad \text{and} \quad U_{PS} := 1 - (1 - p^{l_n})^{p_n}. \]
Lower and upper bounds for reliability
2. Main Limiting Results in EVT

• Let \(X_n = (X_1, \ldots, X_n)\) be a sample of size \(n\) of i.i.d., or even weakly dependent r.v.’s with d.f. \(F\), and let \(X_{i:n}, 1 \leq i \leq n\), denote the associated ascending o.s.’s.

• Extreme Value Theory (EVT) provides a great variety of limiting results that enable us to deal with alternative approaches in the statistical analysis of extreme events.

• The main limiting result in EVT is due to Gnedenko (1943), AM. In this seminal paper, BV Gnedenko has fully characterized the possible non-degenerate limiting distribution of the linearly normalized maximum, \((X_{n:n} - b_n)/a_n, a_n > 0, b_n \in \mathbb{R}\). Such a limit is of the type of the general extreme value (EV) distribution for maxima (EVMD), the unique max-stable law, given by
\[ G(x) \equiv G_\gamma(x) := \begin{cases} 
\frac{e^{-(1+\gamma x)^{-1/\gamma}}}{\gamma}, & 1 + \gamma x > 0 \text{ if } \gamma \neq 0 \\
 e^{-e^{-x}}, & x \in \mathbb{R} \text{ if } \gamma = 0. 
\end{cases} \]

- The shape parameter \( \gamma \), the so-called extreme value index for maxima (EVI\(_M\)), measures the heaviness of the right-tail (or reliability) function \( \overline{F}(x) \equiv R(x) := 1 - F(x) \), as \( x \to +\infty \) and the heavier the right-tail, the larger \( \gamma \) is.

- The EV\(_M\)D is sometimes separated in the three following types,

<table>
<thead>
<tr>
<th>Type</th>
<th>Function</th>
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<tbody>
<tr>
<td>Type I (Gumbel)</td>
<td>( \Lambda(x) = \exp(-\exp(-x)) ), ( x \in \mathbb{R} )</td>
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<tr>
<td>Type II (Fréchet)</td>
<td>( \Phi_\alpha(x) = \exp(-x^{-\alpha}) ), ( x \geq 0 )</td>
</tr>
<tr>
<td>Type III (max-Weibull)</td>
<td>( \Psi_\alpha(x) = \exp(-(-x)^\alpha) ), ( x \leq 0 ),</td>
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</tbody>
</table>

with \( \alpha > 0 \), the types considered in GNEDENKO (1943).
Remark 1. Any result for maxima has its counterpart for minima due to the fact that \(\min_{1\leq i\leq n} X_i = -\max_{1\leq i\leq n} (-X_i)\). We thus have the min-stable laws or \(\text{EV}_m\) D

\[
G^*(x) \equiv G^*_\theta(x) := \begin{cases} 
1 - e^{-(1-\theta x)^{-1}/\theta}, & 1 - \theta x > 0 \quad \text{if } \theta \neq 0 \\
1 - e^{-e^x}, & x \in \mathbb{R} \quad \text{if } \theta = 0.
\end{cases}
\]

- We then say that the d.f. \(F\) of a r.v. \(X\) is in the min-domain of attraction of \(G^*_\theta\), using the notation \(F \in \mathcal{D}_m(G^*_\theta)\), if the d.f. of \(-X\) is in the max-domain of attraction of \(G_\theta\), i.e.

\[
R = 1 - F \in \mathcal{D}_\mathcal{M}(G_\theta).
\]

- The shape parameter \(\theta\), the so-called extreme value index for minima (\(\text{EVI}_m\)), measures the heaviness of the left-tail function \(F(x)\), as \(x \to -\infty\) and the heavier the left-tail, the larger \(\theta\) is.
• Similarly to what happens in the max-scheme, the $\text{EV}_{mD}$ is sometimes separated in the three following types:

<table>
<thead>
<tr>
<th>Type I (min-Gumbel)</th>
<th>$\Lambda^*(x) = 1 - \exp(-\exp(x))$, $x \in \mathbb{R}$</th>
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</thead>
<tbody>
<tr>
<td>Type II (min-Fréchet)</td>
<td>$\Phi^*_\alpha(x) = 1 - \exp(-(\alpha^{-x})^{-\alpha})$, $x \leq 0$</td>
</tr>
<tr>
<td>Type III (Weibull)</td>
<td>$\Psi^*_\alpha(x) = 1 - \exp(-x^\alpha)$, $x \geq 0$.</td>
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**Remark 2.** In most applications involving lifetimes the limit laws $G^*_\theta$ are restricted to the case $\theta \leq 0$. In fact, a lifetime $T$ is always nonnegative. Thus $-T$ is a r.v. with a finite right endpoint and can only be in the max-domain of attraction of a Weibull or a Gumbel. However, since there are systems with large durability, we also often consider the case $\theta > 0$. 
2.1 Rates of convergence and penultimate approximations

- Another important problem in EVT concerns the rate of convergence of $F^n(a_n x + b_n)$ towards $G_\gamma(x)$ or, equivalently, the finding of estimates of the difference

$$d_n(F, G_\gamma, x) := F^n(a_n x + b_n) - G_\gamma(x).$$

- In EVT there exists no analogue of the Berry-Esseen theorem that, under broad conditions, gives a rate of convergence of the order of $1/\sqrt{n}$ in the Central Limit Theorem.

- The rate of convergence depends here strongly on the right-tail of $F$, on the choice of the attraction coefficients, and can be rather slow, as first detected by FISHER & TIPPETT (1928), PCPS. These authors were indeed the first ones to provide a so-called penultimate max-Weibull approximation for $\Phi^n(x)$, with $\Phi$ the normal d.f.
The modern theory of rates of convergence in EVT began with ANDERSON (1971), Contributions to the Asymptotic Theory of Extreme Values, Ph.D. Thesis, University of London. Developments have followed different directions that can be found in a recent paper by BEIRLANT, CAEIRO & GOMES (2012), Revstat.

We refer here only the study of the structure of the remainder $d_n(F, G_\gamma, x)$, with $F \in \mathcal{D}_M(G_\gamma)$, $\gamma \in \mathbb{R}$, i.e. the finding of $d_n \to 0$, as $n \to \infty$, and $\varphi(x)$ such that

$$F^n(a_nx + b_n) - G_\gamma(x) = d_n \varphi(x) + o(d_n).$$

We then say that the rate of convergence of $F^n(a_nx + b_n)$ towards $G_\gamma(x)$ is of the order of $d_n$. 
• In this same framework, the possible **penultimate** behaviour of $F^n(a_nx + b_n)$ has been studied, i.e. the possibility of finding $H(x) = H_n(x)$, perhaps a **max-stable** d.f., such that

$$F^n(a_nx + b_n) - H_n(x) = O(r_n), \quad r_n = o(d_n).$$

• We refer **GOMES & PESTANA** (1987), In L. Puri et al. (eds.), *New Perspectives in Theoretical and Applied Statistics*, Wiley, and **GOMES & DE HAAN** (1999), *Extremes*, who derived, for all $\gamma \in \mathbb{R}$, exact **penultimate approximation rates**, under von Mises-type conditions and some extra differentiability assumptions. **KAUFMANN** (2000), *Extremes*, proved a similar result, but under weaker conditions.

• This **penultimate** or **pre-asymptotic** behaviour has further been studied by **RAOULT & WORMS** (2003), *AAP*, and **DIEBOLT & GUILLOU** (2005), *Revstat*, among others.
Despite of crucial, I am not going to speak about **first** and **second-order** conditions in the field of **extremes**. I just mention that:

- **The first-order conditions** are just necessary and sufficient conditions (or sufficient conditions) to have $F \in D_M(G_\gamma)$.
- **The second-order conditions** essentially measure the rate of convergence in the first-order conditions and depend upon a second-order parameter $\rho \ (\leq 0)$.
- Now we come to the question answered in **GOMES & de HAAN (1999)**: in what circumstances (i.e. for which combination of $\gamma$ and $\rho$) can the convergence rate be improved by the use of penultimate approximations?
- **Answer**: To get any improvement we need to have $\rho = 0$ and to choose $\gamma(t) \equiv \eta(t) = v''(t)/v'(t)$, with $v(t) := u^{-}(t), u(x) := -\ln(-\ln F(x))$. 


3. Ultimate and Penultimate Models for the Sequence of RFs of a Regular & Homogeneous PS System

We first state a theorem in REIS & CANTO E CASTRO (2009), Revstat.

**Theorem 2** (Reis & Canto e Castro, 2009). Any stable law for minima, i.e. $G^*_\theta$, belongs to $\mathcal{D}_M(G_0)$, i.e. there exist sequences \( \{a_n > 0\}_{n \geq 1} \) and \( \{b_n \in \mathbb{R}\}_{n \geq 1} \) such that

\[
(G^*_\theta(a_n x + b_n))^n \xrightarrow{n \to \infty} G_0(x),
\]

uniformly in $\mathbb{R}$. With \( c_n := -\ln(1 - \exp(-1/n)) \), we can choose

\[
a_n = \frac{1}{n(\exp(1/n) - 1)c_n^{\theta+1}}, \quad b_n = \begin{cases} 
\frac{1-c_n^{-\theta}}{\theta} & \text{if } \theta \neq 0 \\
\ln c_n & \text{if } \theta = 0.
\end{cases}
\]
We further state the following result [REIS, CANTO e CASTRO, DIAS & GOMES, 2013, MCAP]:

**Theorem 3.** Let $F \in \mathcal{D}_m(G^*_\theta)$, the min-domain of attraction of $G^*_\theta$, the $\text{EV}_{mD}$, i.e. let us assume that there exist sequences $\{a_n > 0\}_{n \geq 1}$ and $\{b_n \in \mathbb{R}\}_{n \geq 1}$ such that

$$1 - (1 - F(a_n x + b_n))^n \xrightarrow{n \to +\infty} G^*_\theta(x) = 1 - G_\theta(-x),$$

$\forall x \in \mathbb{R}$ and where $G_\theta$ and $G^*_\theta$ are the $\text{EV}_{MD}$ and the $\text{EV}_{mD}$, respectively. Then, for all $\theta \in \mathbb{R}$, and adequate $(l_n, p_n) \to (\infty, \infty)$, as $n \to \infty$, there exist sequences $\{\alpha_n > 0\}_{n \geq 1}$ and $\{\beta_n \in \mathbb{R}\}_{n \geq 1}$ such that for all $t \in \mathbb{R}$,

$$F_n(\alpha_n t + \beta_n) := (1 - (1 - F(\alpha_n t + \beta_n))^{l_n})^{p_n} \xrightarrow{n \to \infty} \Lambda(t) \equiv G_0(t).$$
Consequently, for a regular homogeneous PS system, composed by $p_n$ parallel subsystems with $l_n$ components in series, the sequence of RFs, suitably normalized is such that

$$R_n(\alpha_n t + \beta_n) = 1 - F_n(\alpha_n t + \beta_n) \xrightarrow{n \to \infty} 1 - G_0(t),$$

for all $t \in \mathbb{R}$. We can indeed consider $\alpha_n = a_p n a_p^* n$, $\beta_n = a_p n b_p^* n + b_p n$, where with the same $(a_n, b_n)$ and $c_n$, as before,

$$a_n^* = \frac{1 - \theta b_n^*}{n(\exp(1/n) - 1)c_n}, \quad b_n^* = \begin{cases} \frac{-(l_n/(nc_n)^\theta - 1)}{\theta} & \text{if } \theta \neq 0 \\ -\ln(l_n/(nc_n)) & \text{if } \theta = 0. \end{cases}$$
**Theorem 4.** For all $\theta \neq -1$, a min-stable law $G_\theta^*$ is under the conditions of the main theorem in Gomes & de Haan (1999). Consequently,

$$
\lim_{n \to \infty} \frac{\left(G_\theta^*(a_n x + b_n)\right)^n - G_{\theta_n}(x)}{(\theta + 1)/\ln^2 n} = \frac{x^3 G'_0(x)}{6},
$$

uniformly for all $x \in \mathbb{R}$, with $(a_n, b_n)$ given before and where $\theta_n$ is asymptotically given by

$$
\theta_n = -\frac{\theta + 1}{\ln n} + O\left(\frac{1}{n}\right).
$$

We further have

$$
(G_\theta^*(a_n x + b_n))^n - G_0(x) = O\left(\frac{1}{\ln n}\right).
$$
Remark 3. Note that if $\theta = -1$, von Mises first-order condition holds, but $\rho = -1$ and not zero, as needed for the existence of a penultimate approximation.

If $\theta = -1$, the ultimate approximation $\left( G_{-1}^n (a_n x + b_n) \right)^n \approx G_0 (x)$ cannot be improved, we think, but one never knows . . . .

If $\theta < -1$, $\theta_n > 0 \implies G_{\theta_n}$ is a penultimate sequence of Frechét distributions for $\left( G_{\theta}^* \right)^n$.

If $\theta > -1$, $\theta_n < 0 \implies G_{\theta_n}$ is a penultimate sequence of Weibull distributions for $\left( G_{\theta}^* \right)^n$. 
4. A Small-Scale Simulation Study

We have simulated PS systems with lifetime components from different models, including the $\text{EV}_{mD}(\theta)$, for $\theta = -2(0.5)1$, $p_n = 20, 50, 100$ and $l_n = 2, 10, 20, 50, 100$.

4.1 Testing the EV Condition

$$\mathcal{H}_0 : G_n^* = 1 - (1 - F)^{l_n} \in \mathcal{D}_M(G_\gamma), \text{ for some } \gamma \in \mathbb{R}$$

- We have used the test statistic developed in DIETRICH, DE HAAN & HUESLER (2002), *Extremes* as well as the one in DREES, DE HAAN, LI & PEREIRA (2006), *JSPI*. We have further followed the algorithm proposed in HÜSLER & LI (2006), *Extremes*.

**Conclusions:** $\mathcal{H}_0$ was not rejected and there is no typical behavior on the variation of $l_n$. 
4.2 GoF Test for Gumbel Law

- We have also tested the ultimate law

\[ H_0 : F_n(x) = (1 - (1 - F(x))^{ln})^{pn} = G_0((x - \lambda)/\delta) \]

with \( F_n(x) \) the d.f. of the lifetime of a PS system and \((\lambda, \delta) \in \mathbb{R} \times \mathbb{R}^+\) a vector of unknown (location and scale) parameters.

- For the same set-up described above, we have used the test statistic proposed in LAIO (2004), WRR, not detailed here.

Conclusions

1. The null hypothesis was rejected except for \( \gamma = -1 \) (showing consistency between simulated and theoretical results).
2. Estimated type I error increases as \( \theta \) moves away from \(-1\).
3. Results are not much affected by a variation of \( ln \) (particulary for EV\(_m\) components' lifetimes) and the estimated type I error decreases as \( pn \) increases.
4.3 Gain in Accuracy

- Are the estimates of $\gamma$ closer to the penultimate parameter $\gamma_n$ rather than to the ultimate parameter zero?
- We have considered the theoretical value $\gamma_n = - (\gamma + 1) / \ln n$ and have computed $\hat{\gamma}$ through maximum likelihood (ML).
- On the basis of the $R = 1000$ runs, we have simulated the MSE and BIAS-values,

$$\text{MSE}_P = (1/R) \sum_{i=1}^{R} (\hat{\gamma}_i - \gamma_n)^2, \quad \text{MSE}_U = (1/R) \sum_{i=1}^{R} \hat{\gamma}_i^2,$$

$$\text{BIAS}_P = (1/R) \sum_{i=1}^{R} \hat{\gamma}_i - \gamma_n, \quad \text{BIAS}_U = (1/R) \sum_{i=1}^{R} \hat{\gamma}_i.$$
• In all simulations, we have got for $\gamma \neq -1$ $\text{MSE}_P < \text{MSE}_U$ and $|\text{Bias}_P| < |\text{Bias}_U|$. This leads us to the adoption of **Weibull** or **Fréchet models** for $F_n$.

• **Only** for $\gamma = -1$ were we led to the adoption of a **Gumbel model** for $F_n$. 
5. Conclusions

- The assumption that the number $n$ of system components goes to infinity can be considered somewhat restrictive. Hence the reason for considering a fixed large number of components and pre-asymptotic (penultimate) approximations.

- We are conscious that the restriction that the RF of all components of the system is the same is quite strong, and such an assumption was used only as a simplification.

- More intricate but similar work can be done for non-homogeneous systems.

- Also, an application of the developed theory, out of the scope of this paper, is now under progress.
REFERENCES


THAT’s ALL and THANKS . . .