Extremes for deterministic and random dynamical systems

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The setting

We will consider stochastic processes arising from dynamical systems (both deterministic and randomly perturbed systems). Namely, \( X_0, X_1, X_2, \ldots \) will be such that:

\[
X_n := \varphi \circ T^n = \varphi \circ T \circ \ldots \circ T
\]

where the discrete time dynamical system \((\mathcal{X}, \mathcal{B}, \mathbb{P}, T)\) will denote two different but interrelated settings throughout the paper and \( \varphi : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\} \), is just a measurable observable that achieves a global maximum at some \( \zeta \in \mathcal{X} \).

\( \mathcal{X} \) is a topological space, \( \mathcal{B} \) is the Borel \( \sigma \)-algebra, \( T : \mathcal{X} \rightarrow \mathcal{X} \) is a measurable map and \( \mathbb{P} \) is a \( T \)-invariant probability measure, i.e., \( \mathbb{P}(T^{-1}(B)) = \mathbb{P}(B) \), for all \( B \in \mathcal{B} \).

The \( T \) invariance of \( \mathbb{P} \) implies that \( X_0, X_1, \ldots \) is stationary.
Deterministic dynamics

- $X = \mathcal{M}$ is a compact Riemannian manifold
- $\mathcal{B}$ is the Borel $\sigma$-algebra
- $T = f : \mathcal{M} \to \mathcal{M}$ is a piecewise differentiable map
- $\mathbb{P} = \mu$ is an $f$-invariant probability
- Orbits: $x, f(x), f^2(x) := f(f(x)), \ldots$

$f : [0, 1] \to [0, 1]$

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Random dynamics

- Let $\mathcal{M} = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, for some $d \in \mathbb{N}$ and consider $f : \mathcal{M} \to \mathcal{M}$.
- Let $\theta_\varepsilon$ be a probability supported on $B_\varepsilon(0) := \{ y \in \mathcal{M} : \text{dist}(x, y) < \varepsilon \}$ and such that $\theta_\varepsilon = g_\varepsilon \text{Leb}$ and $0 < \underline{g}_\varepsilon \leq g_\varepsilon \leq \overline{g}_\varepsilon < \infty$.
- For $\omega \in B_\varepsilon(0)$ we define the additive perturbation $f_\omega$ by
  \[ f_\omega(x) = f(x) + \omega. \] (1)
- Let $W_1, W_2, \ldots$ be a sequence of iid r.v. taking values on $B_\varepsilon(0)$, with common distribution given by the probability $\theta_\varepsilon$.
- Let $\Omega = B_\varepsilon(0)^\mathbb{N}$ and $\theta_\varepsilon^\mathbb{N}$ be the product measure defined on $\Omega$.
- Given a point $x \in \mathcal{M}$ and $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, we define the random orbit of $x$ as $x, f_\omega(x), f_\omega^2(x), \ldots$ where:
  \[ f_\omega^n(x) = f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x), \]
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Deterministic representation of random perturbations

Let

\[ S : \mathcal{M} \times \Omega \longrightarrow \mathcal{M} \times \Omega \]

\[(x, \omega) \longmapsto (f_{\omega_1}(x), \sigma(\omega)),\]

where \(\sigma : \Omega \rightarrow \Omega\) is the shift \(\sigma(\omega) = \sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)\).

Hence, the random evolution can fit the original model \((\mathcal{X}, \mathcal{B}, \mathbb{P}, T)\) by taking:

- \(\mathcal{X} = \mathcal{M} \times \Omega\)
- \(\mathcal{B}\) is the respective product \(\sigma\)-algebra
- \(\mathbb{P} = \mu_\varepsilon \times \theta_\varepsilon^\mathbb{N}\)
- the system is then given by the skew product map \(T = S\).
Definition (Decay of correlations)

Let $C_1, C_2$ be two Banach spaces of real functions. We denote the correlation of non-zero functions $\phi \in C_1$ and $\psi \in C_2$ w.r.t. a measure $\mathbb{P}$ as

$$\text{Cor}_{\mathbb{P}}(\phi, \psi, n) := \frac{1}{\|\phi\|_{C_1} \|\psi\|_{C_2}} \left| \int \phi(\psi \circ T^n) d\mathbb{P} - \int \phi d\mathbb{P} \int \psi d\mathbb{P} \right|.$$

We say that we have decay of correlations, w.r.t. $\mathbb{P}$, for observables in $C_1$ against observables in $C_2$ if, for all $\phi \in C_1$, $\psi \in C_2$ we have

$$\text{Cor}_{\mathbb{P}}(\phi, \psi, n) \to 0, \quad \text{as } n \to \infty.$$

We say that we have decay of correlations against $L^1$ observables whenever $C_2 = L^1(\text{Leb})$ and $\|\psi\|_{C_2} = \|\psi\|_1 = \int |\psi| \, d\text{Leb}$. 
Stochastic processes arising from deterministic/random dynamical systems

Let \( \varphi : \mathcal{M} \rightarrow \mathbb{R} \cup \{+\infty\} \).

In the deterministic setting \( X_0, X_1, X_2, \ldots \) is given by

\[
X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}.
\]  

(3)

In the random dynamics case, the process will be

\[
X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N},
\]  

(4)

which can also be written as \( X_n = \bar{\varphi} \circ S^n \), where

\[\bar{\varphi} : \mathcal{M} \times \Omega \rightarrow \mathbb{R} \cup \{+\infty\}, \text{ is given by } \varphi(x, \omega) = \varphi(x).\]

We assume that the \( \varphi : \mathcal{M} \rightarrow \mathbb{R} \cup \{\pm \infty\} \) achieves a global maximum at \( \zeta \in \mathcal{M} \) (we allow \( \varphi(\zeta) = +\infty \)).

We also assume that \( \varphi \) and \( \mathbb{P} \) are sufficiently regular so that, for \( u \) sufficiently high, the event \( U(u) = \{ X_0 > u \} \) corresponds to a topological ball centred at \( \zeta \). Moreover, the quantity \( \mathbb{P}(U(u)) \), as a function of \( u \), varies continuously on a neighbourhood of \( u_F := \sup \{ x : F(x) < 1 \} \), where \( F(x) = \mathbb{P}(X_0 \leq x) \).
Extreme Value Laws

We have an exceedance of the level $u \in \mathbb{R}$ at time $j \in \mathbb{N}$ if the event $\{X_j > u\}$ occurs. Define a new sequence of random variables (r.v.) $M_1, M_2, \ldots$ given by

$$M_n = \max\{X_0, \ldots, X_{n-1}\}.$$ (5)

**Definition**

We say that we have an EVL for $M_n$ if there is a d.f. $H : \mathbb{R} \rightarrow [0, 1]$, with $H(0) = 0$ and, for all $\tau > 0$, there exists a sequence of levels $u_n = u_n(\tau)$, s.t.

$$n \mathbb{P}(X_0 > u_n) \rightarrow \tau, \text{ as } n \rightarrow \infty,$$ (6)

and for which the following holds:

$$\mathbb{P}(M_n \leq u_n) \rightarrow \bar{H}(\tau), \text{ as } n \rightarrow \infty.$$ (7)

When $\bar{H}(\tau) = e^{-\theta \tau}$, where $0 \leq \theta \leq 1$, we say we have an Extremal Index $\theta$. 
An extremal dichotomy

We begin with a dichotomy that was first realised to exist in [FFT12]. The actual statement we present here comes from the paper [AFV12] and uses the results of [FFT13] to cover the case of periodic points.

**Theorem**

Consider a continuous dynamical system \((M, B, \mu, f)\) for which there exists a Banach space \(C\) such that for all \(\phi \in C\) and \(\psi \in L^1(\mu)\), 
\[
\text{Cor}_\mu(\phi, \psi, n) \leq Cn^{-2},
\]
where \(C > 0\) is a constant independent of both \(\phi, \psi\). Let \(X_0, X_1, \ldots\) be given by (3), where \(\varphi\) achieves a global maximum at \(\zeta\). Let \(u_n\) be such that (6) holds. We assume that there exists \(C' > 0\) such that for all \(n\) we have \(\mathbf{1}_{U(u_n)} \in C, \|\mathbf{1}_{U(u_n)}\|_C \leq C'\).

- If \(\zeta\) is a non-periodic point, then there exists an EVL for \(M_n\) and 
  \(H(\tau) = 1 - e^{-\tau}\).

- If \(\zeta\) is a repelling periodic point of prime period \(p\), then there exists an EVL for \(M_n\) and 
  \(H(\tau) = 1 - e^{-\theta \tau}\), where 
  \(\theta = \lim_{n \to \infty} \mathbb{P}(X_0 > u_n, X_p \leq u_n | X_0 > u_n)\).
Smoothing effect of adding noise on the Extremal dichotomy

Theorem ([AFV12])

Consider a dynamical system \((\mathcal{M} \times \Omega, \mathcal{B}, \mu_\varepsilon \times \theta_\varepsilon^N, S)\), where \(\mathcal{M} = \mathbb{T}^d\), for some \(d \in \mathbb{N}\), \(f : \mathcal{M} \to \mathcal{M}\) is a deterministic system which is randomly perturbed as in (1) and \(S\) is the skew product map defined in (2). Assume that there exists \(\eta > 0\) such that \(\text{dist}(f(x), f(y)) \leq \eta \text{dist}(x, y)\), for all \(x, y \in \mathcal{M}\). Assume also that the measure \(\mu_\varepsilon\) is such that \(\mu_\varepsilon = h_\varepsilon \text{Leb}\), with \(0 < h_\varepsilon \leq h_\varepsilon \leq \overline{h}_\varepsilon < \infty\).

Suppose that there exists a Banach space \(\mathcal{C}\) of real valued functions defined on \(\mathcal{M}\) such that for all \(\phi \in \mathcal{C}\) and \(\psi \in L^1(\mu_\varepsilon)\), \(\text{Cor}_{\mu_\varepsilon \times \theta_\varepsilon^N}(\phi, \psi, n) \leq Cn^{-2}\), where \(C > 0\) is a constant independent of both \(\phi, \psi\).

Let \(u_n\) be such that (6) holds and assume that there exists \(C' > 0\) such that for all \(n\) we have \(\mathbf{1}_{U(u_n)} \in \mathcal{C}\), \(\|\mathbf{1}_{U(u_n)}\|_{\mathcal{C}} \leq C'\). For any point \(\zeta \in \mathcal{M}\), consider that \(X_0, X_1, \ldots\) is defined as in (4), then there exists an EVL for \(M_n\) such that \(H(\tau) = 1 - e^{-\tau}\).


Workshop on Risk Analysis and Extreme Values, Paris, 2005
Conference "Extreme Value Theory and Laws of Rare Events"

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